INvariance OF PROJECTIONS IN THE DIAGONAL
OF A NESt ALGEBRA

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ABSTRACT. The study of operator factorization along commutative subspace
lattices which are not nests leads to the investigation of the mapping \( \phi_A \) which
takes an orthogonal projection \( Q \) in the diagonal of a nest algebra \( \mathcal{A} \) to the pro-
jection on the closure of the range of \( AQ \) for certain bounded linear operators
\( A \). The purpose of this paper is to demonstrate that if \( B \) is an operator leaving
the range of \( Q \) invariant, \( V \) is an element of the “Larson radical” of \( \mathcal{A} \), \( B + V \)
is invertible, \( (B + V)^{-1} \) belongs to \( \mathcal{A} \), and \( \phi_{B + V}(Q) \) is in the diagonal of \( \mathcal{A} \),
then \( \phi_V(Q) \leq Q \). For example, if \( V \) is in the Jacobson radical of \( \mathcal{A} \) and \( \lambda \)
is a nonzero scalar, it follows that \( \phi_{\lambda I + V}(Q) = Q \) if and only if \( \phi_{\lambda I + V}(Q) \) belongs
to the diagonal of \( \mathcal{A} \). Examples of the applications to operator factorization
and unitary equivalence of sets of projections are given.

Theorem 1 of [2] gives a fact about the projections in the diagonal of a nest
algebra which is closely related to the main result of this paper. The theorem in
[2] has significant consequences for the theory of nonanticipative representations of
Gaussian random fields, but its applicability to more general problems of operator
factorization relative to a commutative subspace lattice is limited by a technical
hypothesis relating the nest whose diagonal projections are to be studied to an
operator \( A \). The purpose of this investigation is to substitute for that technical
hypothesis the more natural condition that \( A = \lambda I + V \) for some scalar \( \lambda \) and some
\( V \) in the Larson radical of the nest. This substitution broadens the applicability of the
result and strengthens its connections with the existing nest algebra literature.

\( H \) denotes a (real or complex) Hilbert space of any dimension. \( B(H) \) is the
space of all bounded, linear operators on \( H \). For any set \( S \) of projections in \( B(H) \),
alg \( S \) denotes the set of all elements of \( B(H) \) which leave invariant the ranges
of all elements of \( S \). The commutant of \( S \) is called the “diagonal” of \( \text{alg} \, S \) because it
is the intersection of \( \text{alg} \, S \) and the set of adjoints of elements in \( \text{alg} \, S \). For any \( A 
\) in \( B(H) \), \( \text{rp}(A) \) denotes the projection on the closure of \( R(A) \), the range of \( A \). We
use \( \| \cdot \| \) for the norms in \( H \) and \( B(H) \).

In all that follows, \( \Pi \) will denote a chain of projections in \( B(H) \) which contains
the elements 0 and 1. An interval in \( \Pi \) is \( P_2 - P_1 \) where \( P_i \in \Pi \) for \( i = 1, 2 \) and
\( P_1 < P_2 \). A partition of \( \Pi \) is a countable set of mutually orthogonal intervals \( E_a 
\) of \( \Pi \) with \( \sum_a E_a = 1 \). The Larson radical of \( \text{alg} \, \Pi \) is \( \{ A \in \text{alg} \, \Pi : \text{for every } \varepsilon > 0 \)
there exists a partition \( \{ E_a \} \) with \( \sup_a \| E_a A E_a \| < \varepsilon \} \). We refer the reader to
[4 and 5] for further information about the Larson radical, but we need only the
definition here. J. R. Ringrose’s characterization of the Jacobson radical of \( \text{alg} \, \Pi \) for

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a complete chain $\Pi |\{6\}$ motivated Larson's definition and implies that the Jacobson radical is contained in the Larson radical.

**Remark.** Elementary arguments show that $\sup_a \|E_aAE_a\| = \| \sum_a E_aAE_a \|$ (because of the orthogonality of the intervals), even if $A$ depends on $a$.

**Theorem.** Let $\Pi$ be a chain of projections in $B(H)$ with $0$ and $I$ in $\Pi$. Let $Q$ be any projection which commutes with every element of $\Pi$, and let $V$ be any element of the Larson radical of alg $\Pi$. Suppose there exists $B$ in $B(H)$ such that $B$ leaves $R(Q)$ invariant, $B + V$ is invertible, $(B + V)^{-1}$ belongs to alg $\Pi$, and $\rho_p((B + V)Q)$ commutes with $\Pi$. Then $V$ leaves $R(Q)$ invariant. Thus if $V$ is quasinilpotent (in particular, if $V$ is in the Jacobson radical of alg $\Pi$), $\rho_p((\lambda I + V)Q) = Q$ for all nonzero scalars $\lambda$ such that $\rho_p((\lambda I + V)Q)$ commutes with $\Pi$.

**Proof.** Let $A$ denote $B + V$, and choose $f$ in $R(Q)$. Notice that for all $P$ in $\Pi$, $PAf$ belongs to $R(\rho_p(AQ)P)$ because $P$ commutes with $\rho_p(AQ)$.

Given $\varepsilon > 0$, choose a partition of $\Pi$ consisting of intervals $E_a$ such that $\sup_a \| EaVE_a \| < \varepsilon$. Denote the endpoints of $E_a$ as $P_1(a)$ and $P_2(a)$ with $P_1(a) < P_2(a)$.

For all $a$, let $h_a = A^{-1}P_2(a)Af$, $h_a = Qh_a$ because $h_a$ belongs to $A^{-1}(R(\rho_p(AQ)P_2(a))) \subset A^{-1}(R(AQ)) \subset R(Q)$.

Also, $h_a = P_2(a)h_a$ because $A^{-1}$ belongs to alg $\Pi$.

$$(I - Q)Ah_a = (I - Q)P_2(a)Af = P_2(a)(I - Q)(Bf + Vf) = (I - Q)P_2(a)Vf.$$  

Also, $(I - Q)Ah_a = (I - Q)Vh_a$. Equating these two expressions for $(I - Q)Ah_a$, we have $(I - Q)Vh_a = (I - Q)P_2(a)Vf$, which implies

$$E_a(I - Q)Vh_a = E_a(I - Q)Vf \quad \text{for all } a.$$  

For all $a$, $E_a(I - Q)Vh_a = (I - Q)EaVEaP_2(a)h_a = (I - Q)E_aVE_aP_2(a)h_a$ (because $E_a \leq P_2(a)$ and the endpoints of $E_a$ are invariant projections for $V$); thus, $\sum_a E_a(I - Q)Vh_a = \sum_a (I - Q)E_aVE_aP_2(a)h_a$. By $(\ast)$, $\sum_a E_a(I - Q)Vh_a = \sum_a E_a(I - Q)Vf = (I - Q)Vf$. Combining these last two results,

$$\|(I - Q)Vf\| = \left\| \sum_a (I - Q)E_aVE_a h_a \right\| = \left\| \sum_a E_a(I - Q)E_aVE_a A^{-1}P_2(a)Af \right\|$$

$$\leq \left\| \sum_a E_a(I - Q)E_aVE_a A^{-1}P_2(a) \right\| \cdot \| Af \|$$

$$= \left\| \sum_a E_a(I - Q)E_aVE_a A^{-1}E_a \right\| \cdot \| Af \|$$

(because $A^{-1}$ maps $R(P_1)$ into $R(P_1)$, so $E_aA^{-1}P_1(a) = 0$)

$$= \sup_a \| E_a(I - Q)E_aVE_a A^{-1}E_a \| \cdot \| Af \|$$

(by the remark preceding the theorem)

$$\leq \sup_a \| E_aVE_a \| \cdot \| A^{-1} \| \cdot \| Af \| < \varepsilon \| A^{-1} \| \cdot \| Af \|.$$
The arbitrariness of $\varepsilon$ now implies that $(I - Q)Vf = 0$ for all $f$ in $R(Q)$. That is, $R(Q)$ is invariant under $V$.

The final assertion of the theorem follows from the invariance of $R(Q)$ under $\kappa I + V$ and $(\kappa I + V)^{-1}$, the latter having a convergent power series expansion in terms of $V$. \hfill \Box

To illustrate the importance of the preceding theorem to the operator factorization problem, we state below a corollary which is easily derived from the preceding theorem using the following observation from [1, p. 405]. Let $S$ be a selfadjoint, positive-definite, bounded, linear operator on $H$. Let $X$ and $Y$ be elements of $B(H)$. The following conditions are equivalent:

(i) There exists $A$ in $B(H)$ such that $A^*A = S$ and $\text{rp}(AX)$ commutes with $\text{rp}(AY)$.

(ii) $\text{rp}(AX)$ commutes with $\text{rp}(AY)$ for all $A$ such that $A^*A = S$.

Let us use the term bordered chain for a chain which contains 0 and $I$.

**Corollary 1.** Let $T$ be an invertible element of $B(H)$, and let $\Pi$ be any bordered chain of projections. Suppose that $T^*T = (\lambda I + V)^*(\lambda I + V)$ for some scalar $\lambda$ and element $V$ of the Larson radical of alg $\Pi$ such that $(\lambda I + V)^{-1}$ belongs to alg $\Pi$. Then for any projection $Q$ in the commutant of $\Pi$, if $\text{rp}(TQ)$ commutes with $\Pi$, then $\text{rp}((\lambda I + V)Q) < Q$.

Assume from now on that $H$ is separable.

Observe that if the operator $T$ of Corollary 1 is invertible and satisfies $T^*T = I + K$ with $K$ in the Macaev ideal, then the well-known factorization theory of Gohberg and Krein [3] may be applied. (In particular, this includes the case that $K$ is a Hilbert-Schmidt operator, which arises in the theory of nonanticipative representations of Gaussian random fields.) If $\Pi$ is a complete, bordered, continuous chain, then a factorization $T^*T = (I + V)^*(I + V)$ with $V$ a Volterra operator in the Jacobson radical of alg $\Pi$ is provided by the theory of [3]. In this case, $(I + V)^{-1}$ belongs to alg $\Pi$ and the invariance of $R(Q)$ under $V$ is equivalent to $\text{rp}((I + V)Q) = Q$. Thus we have

**Corollary 2.** Let $S$ be a commutative set of projections in $B(H)$ which contains a complete, continuous, bordered chain $\Pi$. Let $T$ in $B(H)$ be invertible and satisfy $T^*T = I + K$ with $K$ in the Macaev ideal. Then the Gohberg-Krein factorization of $T^*T$ along $\Pi$, $T^*T = (I + V)^*(I + V)$, satisfies $\text{rp}((I + V)P) = P$ for all $P$ in $S$ if and only if $\{\text{rp}(TP): P \in S\}$ is commutative.

Finally, the standard technique of working with the unitary operator $T(I + V)^{-1}$ translates operator factorization results to results about unitary equivalence. (Two sets of projections, $S$ and $T$, are unitarily equivalent if there exists a unitary operator $U$ such that $T = \{UPU^*: P \in S\}$.)

**Corollary 3.** Let $S$ and $T$ be as in Corollary 2. Then $\{\text{rp}(TP): P \in S\}$ is unitarily equivalent to $S$ if and only if it is commutative.

**References**


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