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LAGRANGIAN SYSTEMS IN THE PRESENCE OF SINGULARITIES

A. CAPOZZI, C. GRECO AND A. SALVATORE

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ABSTRACT. In this paper we study dynamical systems embedded in a conservative field of forces, whose potential is "singular." We look for T-periodic solutions of these systems by variational methods.

0. Introduction. In this paper we look for T-periodic solutions of the Lagrangian system of ordinary differential equations:

$$(*)\qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(t,q,\dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(t,q,\dot{q}) = 0, \qquad q \in C^2(\mathbf{R},\mathbf{R}^N),$$

where the Lagrangian function $\mathcal{L}(t,q,\xi)$ is given, as usual, by

$$\mathcal{L}(t,q,\xi) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(t,q)\xi_i\xi_j + \sum_{i=1}^{N} b_i(t,q)\xi_i + c(t,q) - V(t,q),$$
$$t \in \mathbf{R},\ q,\xi \in \mathbf{R}^N,$$

and $a_{ij}(t,q)$, $b_i(t,q)$, $c(t,q)$, $V(t,q)$ are C^1 real-valued functions, T-periodic in t. Moreover we suppose that the "potential" $V(t,q)$ is defined in $\mathbf{R}\times\Omega$, where Ω is an open subset of $\mathbf{R}^N$, and $V(t,q) \to -\infty$ as $q \to \partial\Omega$.

Many authors have studied this problem in the case when $\Omega = \mathbf{R}^N$ (so $\partial\Omega = \emptyset$) under various assumptions on the growth of $V(t,q)$ as $|q| \to \infty$: cf., for instance, [**2**, **3**, **5**, **9**, **10**]. W. B. Gordon was the first to study our case by means of variational methods, and we refer to [**6**, **7**] for the physical motivation of the problem (cf. also the end of this section). Finally we refer to [**1**, **8**] for the case $V(t,q) \to +\infty$ as $q \to \partial\Omega$ ($\Omega \neq \mathbf{R}^N$).

In this paper we suppose that:

(0.1) $\{a_{ij}(t,q)\}_{i,j}$ is a symmetric matrix, and there exists a function $\lambda : \mathbf{R}^N \to]0,+\infty[$ such that:

(i) $\sum_{i,j=1}^N a_{ij}(t,q)\xi_i\xi_j \geq \lambda(q)|\xi|^2$ for any $t\in\mathbf{R}$, $q,\xi\in\mathbf{R}^N$;

(ii) there are real constants $c_1 > 0$ and $\nu \in [0,1[$ such that $\lambda(q) \geq c_1(|q|^\nu + 1)^{-1}$ for any $q \in \mathbf{R}^N$.

(0.2) There exists $M > 0$ such that $|b_i(t,q)| \leq M$ for any $i = 1,2,\ldots,N$ and $t\in\mathbf{R}$, $q\in\mathbf{R}^N$.

(0.3) $c(t,q) \geq 0$ for any $t\in\mathbf{R}$, $q\in\mathbf{R}^N$.

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(0.4) There exist a function $U \in C^1(\Omega, \mathbf{R})$, a neighborhood $\mathcal{N}$ of $\mathbf{R}^N \backslash \Omega$ and a constant $c_2 \geq 0$ such that:
(i) $\lim_{q \to \partial\Omega} U(q) = -\infty$;
(ii) $-V(t,q) \geq |U'(q)|^2 - c_2$ for any $t \in \mathbf{R}$ and $q \in \mathcal{N} \cap \Omega$.

The theorems we now state were announced in rather different form in [**4**] (cf. Remark 0.2 for some generalizations).

THEOREM 0.1. *Let $N = 2$ and $\Omega = \mathbf{R}^2 \backslash \{0\}$. Assume that* (0.1)–(0.4) *hold, and*

(0.5) *there exist real constants $c_3, c_4 > 0$ and $\mu \in [0, 2 - \nu[$ such that $V(t,q) \leq c_3|q|^\mu + c_4$ for any $t \in \mathbf{R}$, $q \in \Omega$.*

Then there exists at least one T-periodic solution of $(*)$.

THEOREM 0.2. *Let $N \geq 2$, and let Ω be symmetric with respect to the origin. Assume that* (0.1)–(0.5) *hold, and*

(0.6) $$a_{ij}, b_i, c, \text{ and } V \text{ are } (T/2)\text{-periodic in } t;$$

(0.7) $$a_{ij}, c, \text{ and } V \text{ are even in } q \text{ and } b_i \text{ are odd in } q.$$

Then there exists a pair $(q, -q)$ of T-periodic solutions of $(*)$.

REMARK 0.1. If $b_i = 0$ and we deal with the autonomous case, then under the same hypotheses of Theorem 0.1 or 0.2 there exists a solution of $(*)$ with minimal period T (cf. §1 below).

Notice that hypotheses (0.1) and (0.2) are motivated by the positiveness of the kinetic energy of a mechanical system; moreover we emphasize that $\{a_{ij}(t,q)\}_{i,j}$ can go to zero as $|q| \to \infty$. Condition (0.4) has been introduced by W. B. Gordon (cf. [**6, 7**]); in [**6**] he considers the autonomous second-order Hamiltonian system: $\ddot{q} = -V'(q)$ in the case when $V(q)$ is bounded from above as $|q| \to \infty$.

We observe that condition (0.5) permits one to consider, for instance, conservative dynamical systems with a periodic "forcing" term.

REMARK 0.2. As we shall see in the sequel, Theorem 0.1 still holds under more general assumptions on the open set Ω: for example, in the case when $\Omega = \mathbf{R}^N \backslash P_1 \cup P_2$, where $N > 2$ and P_i $(i = 1, 2)$ are linear subspaces of $\mathbf{R}^N$ such that $P_1 \cap P_2 = \{0\}$ and $1 \leq \dim(P_i) \leq N - 2$, $i = 1, 2$. This is the geometrical situation which occurs in the study of the planar n-body problem (cf. [**6**]).

Finally we give a simple application of Theorem 0.1. Let us consider the system: $\ddot{q} = |q|^{-2} + f(t)$ where $f\colon \mathbf{R} \to \mathbf{R}^2$ is continuous and periodic. If we set $V(t,q) = -|q|^{-2} + (f(t)|q)_{\mathbf{R}^2}$ then (0.1)–(0.5) are satisfied; (0.4) in particular, holds with $U(q) = \log|q|$, $\mathcal{N} = \{q \in \mathbf{R}^2 |\ |q| < 1\}$, and $c_2 = \|f\|_\infty$.

1. Proofs of the theorems.

If $x, y \in \mathbf{R}^N$, we denote by $|x|$ and xy the Euclidean norm and the inner product in $\mathbf{R}^N$. For $1 \leq p \leq \infty$ let $\|q\|_p$ be the norm in $L^p(\mathbf{R}, \mathbf{R}^N)$; moreover we consider the Sobolev space $H^1 = H^{1,2}([0,T], \mathbf{R}^N)$ obtained by the closure of the C^∞ T-periodic functions $q(t)$ with respect to the norm

$$\|q\| = \left(\int_0^T |q(t)|^2\, dt + \int_0^T |\dot{q}(t)|^2\, dt \right)^{1/2}.$$

H^1 is an Hilbert space, and we denote by $\langle\cdot,\cdot\rangle$ the duality between H^1 and its dual H^{-1}. We recall that H^1 is compactly embedded in $C([0,T],\mathbf{R}^N)$.

Finally, we set $\mathrm{Im}(q)=\{q(t)|t\in[0,T]\}$ and $\delta(q)=\sup_{x,y\in\mathrm{Im}(q)}|x-y|$.

Let us consider the functional:

$$f(q)=\frac{1}{2}\int_0^T\sum_{i,j=1}^N a_{ij}(t,q)\dot q_i\dot q_j\,dt+\int_0^T\sum_{i=1}^N b_i(t,q)\dot q_i\,dt$$
$$+\int_0^T c(t,q)\,dt-\int_0^T V(t,q)\,dt,$$

which is defined on the open subset $\Lambda^1\Omega\equiv\{q\in H^1|\mathrm{Im}(q)\subset\Omega\}$ of H^1. It is easy to verify that $f\in C^1(\Lambda^1\Omega,\mathbf{R})$ and its critical points (that is, the zeros of f') are T-periodic solutions of $(*)$. The functional f is, in general, not bounded from below on $\Lambda^1\Omega$; then, in order to prove Theorem 0.1, we shall check that f attains its minimum value on a nontrivial homotopy class of $\Lambda^1\Omega$, where $\Omega=\mathbf{R}^2\backslash\{0\}$.

We shall need the following lemma (in the sequel $a_1,a_2,\dots$ will denote positive constants).

LEMMA 1.1. *Let $\Omega=\mathbf{R}^2\backslash\{0\}$. Then $\|q\|_\infty\le\delta(q)$ for any $q\in\Lambda^1\Omega$ which is not homotopic to a constant in Ω.*

PROOF. We argue by contradiction and assume that $|q(t_0)|>\delta(q)$ for some $t_0\in[0,T]$. Then $\mathrm{Im}(q)$ is contained in $\{x\in\mathbf{R}^2|\ |x-q(t_0)|\le\delta(q)\}\subset\Omega$, and q is homotopic to a constant in Ω, so we get a contradiction.

REMARK 1.1. Since $|q(t_1)-q(t_2)|=|\int_{t_1}^{t_2}\dot q(t)\,dt|\le\|\dot q\|_2|t_1-t_2|^{1/2}$, Lemma 1.1 implies: $\|q\|\le c_5\|\dot q\|_2$, where $c_5>0$.

PROOF OF THEOREM 0.1. Let Λ_0 be a nontrivial homotopy class. In our assumptions, for any $q\in\Lambda_0$, we have

$$\begin{aligned}f(q)&=\frac{1}{2}\int_0^T\sum_{i,j=1}^N a_{ij}(t,q)\dot q_i\dot q_j\,dt+\int_0^T\sum_{i=1}^N b_i(t,q)\dot q_i\,dt\\&\quad+\int_0^T c(t,q)\,dt-\int_0^T V(t,q)\,dt\\&\ge\frac{1}{2}\int_0^T\lambda(q)|\dot q|^2\,dt-M\int_0^T|\dot q|\,dt-\int_0^T(c_3|q|^\mu+c_4)\,dt\\&\ge\frac{1}{2}\int_0^T c_1(|q|^\nu+1)^{-1}|\dot q|^2\,dt-M\|\dot q\|_1-c_3\|q\|_\mu^\mu-Tc_4\\&\ge\frac{c_1}{2}(a_1\|q\|^\nu+1)^{-1}\|\dot q\|_2^2-M\|\dot q\|_1-c_3\|q\|_\mu^\mu-Tc_4\\&\ge\frac{c_1}{2}(a_1\|q\|^\nu+1)^{-1}\frac{1}{c_5^2}\|q\|^2-a_2\|q\|-a_3\|q\|^\mu-Tc_4\ge-a_4\end{aligned}$$

because $2-\nu>\max\{1,\mu\}$.

So the functional f is bounded from below on Λ_0. Let $(q_n)_n\subset\Lambda_0$ be a minimizing sequence of f. From coercivity, it is bounded in H^1; then we can select a subsequence (still denoted by $(q_n)_n$) such that $q_n\to q_0\in H^1$ weakly in H^1 and uniformly. We shall prove that, if $q_0 \in \partial\Lambda^1\Omega$, then we have

$$\lim_{n\to\infty} \int_0^T V(t, q_n)\, dt = -\infty, \tag{1.1}$$

and this is not possible because $(f(q_n)_n)$ is bounded.

In fact, suppose $q_0 \in \partial\Lambda^1\Omega$; since if $\operatorname{Im}(q_0) \subset \partial\Omega$ (1.1) is trivial, we can assume that $q_0(0) \in \partial\Omega$ and there exists $\tau > 0$ such that $q_0(t) \in \mathcal{N} \cap \Omega$ for any $t \in]0, \tau]$. Since $(\|\dot{q}_n\|_2)_n$ and $(U(q_n(\tau)))_n$ are bounded, we have

$$\begin{aligned} -a_5 - U(q_n(0)) &\le U(q_n(\tau)) - U(q_n(0)) \\ &= \int_0^\tau \frac{d}{dt} U(q_n(t))\, dt \le \int_0^\tau |U'(q_n)|\, |\dot{q}_n|\, dt \\ &\le \left(\int_0^\tau |U'(q_n)|^2\, dt\right)^{1/2} \|\dot{q}_n\|_2 \le a_6 \left(\int_0^\tau |U'(q_n)|^2\, dt\right)^{1/2}, \end{aligned}$$

and therefore, by (0.4)(i),

$$\lim_{n\to\infty} \int_0^\tau |U'(q_n)|^2\, dt = +\infty. \tag{1.2}$$

Since $V(t, q_n(t)) \le a_7$ for any $t \in [0, T]$ and $n \in \mathbf{N}$, by (0.4)(ii) it follows that

$$\begin{aligned} \int_0^T V(t, q_n)\, dt &= \int_0^\tau V(t, q_n)\, dt + \int_\tau^T V(t, q_n)\, dt \\ &\le -\int_0^\tau |U'(q_n)|^2\, dt + \tau c_2 + a_7(T - \tau). \end{aligned}$$

By the previous inequality and (1.2) we get (1.1).

We can conclude that q_0 belongs to $\Lambda^1\Omega$ and to Λ_0 also (by the uniform convergence). Moreover, by the weakly lower semicontinuity of f, it follows that q_0 is the minimum of f on Λ_0.

Let us consider now the case $N \ge 2$ (as in Theorem 0.2). Since Ω can be homotopically trivial (for example if $\Omega = \mathbf{R}^3 \setminus \{0\}$), we shall restrict the functional f to the subspace (cf. [**3**])

$$E = \{q \in H^1 | q(t + T/2) = -q(t)\}.$$

Observe that each $q \in E$ has mean value zero; then Wirtinger's inequality holds:

$$c_6 \|q\|^2 \le \|\dot{q}\|_2^2 \quad \text{for any } q \in E, \qquad \text{where } c_6 = 2\pi/(T + 2\pi). \tag{1.3}$$

The choice of the subspace E is also motivated by the following lemma.

LEMMA 1.2. *If $q \in E \cap \Lambda^1\Omega$ is a critical point of $f_{|E\cap\Lambda^1\Omega}$, then q is a critical point of f.*

PROOF. Let $q \in E \cap \Lambda^1\Omega$ be a critical point of $f_{|E\cap\Lambda^1\Omega}$, that is,

$$\langle f'(q), h\rangle = 0 \quad \text{for any } h \in E. \tag{1.4}$$

We prove that (1.4) holds for any $h \in H^1$. In order to get this, fix $h \in H^1$ and set $h(t) = h_1(t) + h_2(t)$, where

$$h_1(t) = h(t) - h(t + T/2), \quad \text{and} \quad h_2(t) = h(t + T/2).$$

Since $h_1 \in E$, by (1.4), we have

$$0 = \langle f'(q), h_1 \rangle = \langle f'(q), h \rangle - \langle f'(q), h_2 \rangle.$$

By (0.6), (0.7) it is easy to verify that $\langle f'(q), h_2 \rangle = -\langle f'(q), h \rangle$. Then the conclusion follows.

PROOF OF THEOREM 0.2. By Lemma 1.2 it suffices to look for the critical points of the functional f on the open subset $E \cap \Lambda^1\Omega$ of E. By using (1.3) instead of Lemma 1.1, it can be seen, as in the proof of Theorem 0.1, that f is bounded from below and attains its minimum value.

Let us consider the autonomous case, and assume that $b_i(q) = 0$ $(i = 1, 2, \dots, N)$. Moreover, suppose that q is a critical point of f and T/k $(k \in \mathbf{N}, k \geq 2)$ is a period of the function $q(t)$. Then it is easy to check that $q_k(t) = q(t/k)$ is still a critical point of f, and that $f(q_k) < f(q)$ (cf. [**3**]). Therefore T is the minimal period of the solutions given by Theorems 0.1 and 0.2.

Now we shall sketch some generalizations of Theorem 0.1, as mentioned in Remark 0.2. Let Ω be an open set of $\mathbf{R}^N$ $(N \geq 2)$. As in [**6**], we consider the following subset $\Lambda \subset \Lambda^1\Omega$:

> $q \in \Lambda \Leftrightarrow$ for any $c > 0$ there exists a compact subset K_c of $\mathbf{R}^N$ which contains any $p \in \Lambda^1\Omega$ such that p is homotopic to q in Ω and the arclength of p is $\leq c$.

Clearly, if $\Omega = \mathbf{R}^2 \backslash \{0\}$, then $q \in \Lambda \Leftrightarrow q$ is not homotopically trivial, but, in general, this is not the case.

Suppose $\Lambda \neq \emptyset$, and let Λ_0 be a homotopy class. Then we can minimize f on Λ_0 provided a geometrical estimate like Lemma 1.1 is available for the functions in Λ_0. We limit ourself to stating the following variant of Lemma 1.1.

LEMMA 1.3. *Let* $\Omega = \mathbf{R}^N \backslash P_1 \cup P_2$ *when* P_1, P_2 *are as in Remark* 0.2.
Then $\Lambda \neq \emptyset$ *and, if* Λ_0 *is a homotopy class* $\subset \Lambda$, *there exists* $c_7 > 0$ *such that, for any* $q \in \Lambda_0$, *we have* $\|q\|_\infty \leq c_7 \delta(q)$.

PROOF. It is easy to see that $\Lambda \neq \emptyset$; let Λ_0 be a homotopy class $\subset \Lambda$. Now, for any $r > 0$, we set $\rho(r) = \sup\{\sigma > 0 \mid \text{for any } x \in \mathbf{R}^N \text{with } |x| = r,\ P_1 \cap B_\sigma(x) = \emptyset \text{ or } P_2 \cap B_\sigma(x) = \emptyset\}$ $(B_\sigma(x) \equiv \{y \in \mathbf{R}^N \mid |y - x| < \sigma\})$. Then $\rho(r)$ increases proportionally to r, that is, there exists $c_8 > 0$ such that $\rho(r) = c_8 r$. Set $c_7 = 2/c_8$, and suppose that there exists $q \in \Lambda_0$ such that $\|q\|_\infty > c_7\delta(q)$. Then $\|q\|_\infty = |q(t_0)|$ for some $t_0 \in [0, T]$. Set $A = B_{\delta(q)}(q(t_0))$ and observe that $\delta(q) \geq \rho(|q(t_0)|)$ (otherwise $P_1 \cap A = \emptyset$ or $P_2 \cap A = \emptyset$, and this implies $q \notin \Lambda_0$), so $|q(t_0)| = \|q\|_\infty > c_7\delta(q) \geq c_7\rho(|q(t_0)|) = c_7 c_8 |q(t_0)| = 2|q(t_0)|$, which is not possible.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI, CAMPUS UNIVERSITARIO, 70125 BARI, ITALY