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# CYCLIC VECTORS IN $A^{-\infty}$

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ABSTRACT. If $f$ is in $A^{-p}$, then $f$ is cyclic in $A^{-\infty}$ if and only if $f$ is cyclic in every $A^{-q}$ $(q > p)$. An analogous result holds for the Bergman spaces $B^p$.

In this note we apply the theory developed in [**2** and **3**] to explain the relationship between cyclic vectors in $A^{-\infty}$ and $A^{-p}$ or $B^p$.

DEFINITIONS. 1. $A^{-p}$ $(p > 0)$ is the Banach space of analytic functions $f(z)$ in $U = \{z \in C \mid |z| < 1\}$ that satisfy $|f(z)| = o[(1 - |z|)^{-p}]$ $(|z| \to 1)$ with the norm $\|f\| = \max\{|f(z)|(1 - |z|)^p\}$ $(z \in U)$. Note that $f_n \to f$ in $A^{-s}$ and $g_n \to g$ in $A^{-t}$ implies $f_n g_n \to fg$ in $A^{-(s+t)}$. Also one can show that if $f_n(z) \neq 0$, $z \in U$, $f(0) = 1$, then $f_n^\alpha \to f^\alpha$ in $A^{-\alpha s}$ $(0 < \alpha < \infty)$.

2. $B^p$ $(p > 0)$ is the Bergman space, i.e., the "analytic" subspace of $L^p(r\,dr\,d\theta)$ in $U$.

3. $A^{-\infty} = UA^{-p} = UB^p$ $(p > 0)$, $A^{-\infty}$ is a linear topological space (see [**2**, p. 189]); it is the inductive limit of $A^{-p}$.

4. $\mathcal{P}$ is the set of all algebraic polynomials $P(z)$. $\mathcal{P}$ is dense in any of the spaces $A^{-p}, B^p, A^{-\infty}$.

5. Let $A$ be any of the spaces $A^{-p}, B^p, A^{-\infty}$, and let $f \in A$. The subspace generated by $f$ in $A$ which is invariant under the operator of muiltiplication by $z$ on $A$ is

$$I(f; A) = \operatorname{clos}\{fP \mid P \in \mathcal{P}\} = \operatorname{clos}(fg \mid g \in H^\infty).$$

6. An $f \in A$ is called cyclic in $A$ if $I(f; A) = A$.

THEOREM. *If $f \in A^{-p}$ and is cyclic in $A^{-\infty}$ then $f$ is cyclic in every $A^{-q}$ $(q > p)$.*

A nonvanishing $f$ in $A^{-\infty}$ has a representation in terms of a bounded premeasure $\mu$ (see Proposition 4.1 in [**3**]). If $f$ is cyclic, $\mu_\sigma$, the $\kappa$-singular part of the premeasure $\mu$ is 0 and the hypothesis of Corollary 3.1.1 in [**3**] is satisfied. Thus we have the following result.

PROPOSITION. *If $f$ is cyclic in $A^{-\infty}$ then there exists a sequence of functions $\{g_m(z)\}_1^\infty$, each belonging to $A^{-\infty}$, such that*

(a) $g_m(z) \neq 0$ $(z \in U; m = 1, 2, \ldots)$.

(b) $h_m = fg_m$ $(m = 1, 2, \ldots)$ *belongs to $A^{-s}$ for some fixed $s > 0$.*

(c) $\|1 - h_m\|_{A^{-s}} \to 0$ $(m \to \infty)$.

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We need the following

LEMMA. *If* $f \in A^{-p}$, $g \in A^{-\infty}$, $f(z)g(z) \neq 0$ *for* $z \in U$ *and* $h = fg \in A^{-s}$ $(s > p)$ *then* $h \in I(f; A^{-s})$.

PROOF OF THE LEMMA. Let $\phi(\alpha) = fg^{\alpha} = f^{1-\alpha}h^{\alpha} \in A^{-((1-\alpha)p+\alpha s)} \subseteq A^{-s}$. Let $F = \{\alpha | \ 0 \leq \alpha \leq 1, \ \phi(\alpha) \in I(f, A^{-s})\}$. $F$ is closed because $\phi$ is a continuous function from $[0,1]$ to $A^{-s}$. Since $g \in A^{-\infty}$, there is an integer $n$ so that $g \in A^{-n}$. If $\alpha_0 \in F$ and $\alpha_0 < 1$ we will show that $\alpha_0 + \varepsilon \in F$ for

$$\varepsilon < \frac{1}{n}[s - (\alpha_0 s + (1-\alpha_0)p)] = \frac{(1-\alpha_0)(s-p)}{n}.$$

Since $g^{\varepsilon} \in A^{-(1-\alpha_0)(s-p)}$ there exists a sequence of polynomials $P_n$ such that $P_n \to g^{\varepsilon}$ in $A^{-(1-\alpha_0)(s-p)}$. Since $P_n f g^{\alpha_0} \in I[f; A^{-s}]$ and $P_n f g^{\alpha_0} \to f g^{\alpha_0+\varepsilon} = \phi(\alpha_0 + \varepsilon)$ in $A^{-s}$, we have $\phi(\alpha_0 + \varepsilon) \in I(f; A^{-s})$. Since $0 \in F$ we have $F = [0,1]$ and $\phi(1) = h \in I[f, A^{-s}]$.

Note that the Proposition and the Lemma imply that $f$ is cyclic in $A^{-s}$.

PROOF OF THE THEOREM. Let $\{g_m\}_1^{\infty}$ be as in the Proposition with $g_m(0) = 1$. If $s \leq p$ then we have $f$ cyclic in $A^{-p}$. Let $s > p$ and assume $f(0) = 1$. Given $q > p$ we choose an integer $n$ such that $s/n < q - p$. By finite induction we show that $f^{1-k/n}$ $(k = 0, 1, \ldots, n)$ is in $I[f; A^{-q}]$. If $k < n$ and $f^{1-k/n} \in I[f; A^{-q}]$ then

$$f^{1-k/n} g_m^{1/n} = f^{1-(k+1)/n} h_m^{1/n} \in A^{-(p+s/n)} \subset A^{-q}.$$

By the Lemma, $f^{1-k/n} g_m^{1/n} \in I[f^{1-k/n}, A^{-q}] \subset I[f, A^{-q}]$. Since $h_m^{1/n} \to 1$ in $A^{-s/n}$, and $f^{1-(k+1)/n} \in A^{-p}$,

$$\lim_{m\to\infty} f^{1-k/n} g_m^{1/n} = \lim_{m\to\infty} f^{1-(k+1)/n} h_m^{1/n} = f^{1-(k+1)/n} \quad \text{is in } A^{-q},$$

and we have $f^{1-(k+1)/n} \in I(f; A^{-q})$. Thus we have $f^0 = 1 \in I(f, A^{-q})$, i.e., $f$ is cyclic in $A^{-q}$.

COROLLARY TO THE LEMMA. *If* $f \in A^{-p}$ *is invertible in* $A^{-\infty}$, *i.e.*, $|f(z)| \geq c(1-|z|)^{\delta}$ *for some positive* $\delta$, *then* $f$ *is cyclic in every* $A^{-q}$ $(q > p)$.

We remark that the question whether $f$ is cyclic (or invertible) in $A^{-\infty}$ implies $f$ cyclic in $A^{-p}$ is still an open question. This question was first posed by H. S. Shapiro (see Theorem 5 in [**4**] or the remark following Theorem A in [**1**]).

We also note that a result analogous to our theorem can be proven in a similar manner for the spaces $B^p$ $(p > 0)$ and that the corresponding corollary to the lemma for the spaces $B^p$ can be shown to be equivalent to Theorem 5 in [**4**].

## REFERENCES


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