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CONFORMALLY NATURAL EXTENSION OF VECTOR FIELDS FROM S^{n-1} TO B^n

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ABSTRACT. Up to multiplication by a constant there is exactly one conformally natural continuous linear map from the space of continuous vector fields on S^{n-1} to the space of continuous vector fields on B^n.

1. Introduction. For any $n \geq 2$ let G_n be the group of all Möbius transformations of $\mathbf{R}^n \cup \{\infty\}$ that map the open unit ball B^n onto itself. The group G_n also preserves the unit sphere S^{n-1}, so it acts on the vector spaces $\mathcal{T}(S^{n-1})$ and $\mathcal{T}(B^n)$ of continuous vector fields on S^{n-1} and B^n. That action is given by the formula

$$(g \cdot f)(g(x)) = g'(x)f(x), \qquad g \in G_n. \tag{1.1}$$

Here $g'(x)$ is the derivative of the map $g\colon \mathbf{R}^n \cup \{\infty\} \to \mathbf{R}^n \cup \{\infty\}$ at the point x in S^{n-1} or B^n, and the vector field f is interpreted as a continuous map into $\mathbf{R}^n$. If $f \in \mathcal{T}(S^{n-1})$, then $x \cdot f(x) = 0$ for all x in S^{n-1}.

The compact open topology makes $\mathcal{T}(S^{n-1})$ and $\mathcal{T}(B^n)$ topological vector spaces. We are interested in continuous linear maps $L\colon\ \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$ that are conformally natural (G_n-equivariant) in the sense that

$$L(g \cdot f) = g \cdot L(f), \qquad f \in \mathcal{T}(S^{n-1}),\ g \in G_n. \tag{1.2}$$

One such map $L_0\colon\ \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$ is given by

$$(g \cdot L_0(f))(0) = \int_{S^{n-1}} (g \cdot f)(u)\, d\omega(u), \qquad f \in \mathcal{T}(S^{n-1}),\ g \in G_n. \tag{1.3}$$

(Here ω is the rotation invariant Borel measure of mass one on S^{n-1}.) Thurston [**6**, Chapter 11] used a multiple of L_0 in dimension $n = 3$ to define extensions of quasiconformal maps from S^2 to B^3. A certain Dirichlet problem led Ahlfors [**1**, **2**] to study a multiple of L_0 in every dimension $n \geq 2$. H. M. Reimann [**5**] noticed the connection between the results of Thurston and Ahlfors and used Ahlfors' results to extend certain quasiconformal maps from S^{n-1} to B^n.

A new way to extend quasiconformal maps of small dilatation from S^{n-1} to B^n was introduced in [**3**]. Application of that method to small deformations of the identity leads to a continuous conformally natural map $L\colon\ \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$, as we shall see in the next section. Computation shows that L is a multiple of L_0. That is no accident. In the final section of this paper we shall prove the following theorem.

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THEOREM 1. *Every conformally natural continuous linear map from $\mathcal{T}(S^{n-1})$ to $\mathcal{T}(B^n)$ is a multiple of L_0.*

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2. Infinitesimal barycentric extensions. The barycentric extension method in [**3**] extends any homeomorphism $\varphi\colon S^{n-1} \to S^{n-1}$ to a continuous map $\Phi = \operatorname{ex}(\varphi)\colon \operatorname{Cl}(B^n) \to \operatorname{Cl}(B^n)$ in a conformally natural way:

$$\operatorname{ex}(g \circ \varphi \circ h) = g \circ \operatorname{ex}(\varphi) \circ h, \qquad g, h \in G_n. \tag{2.1}$$

If $f \in \mathcal{T}(S^{n-1})$ is smooth there is a one-parameter group of diffeomorphisms $\varphi_t\colon S^{n-1} \to S^{n-1}$ such that near $t = 0$

$$\varphi_t(u) = u + tf(u) + o(t), \qquad u \in S^{n-1}, \tag{2.2}$$

uniformly in u. Let $\Phi_t = \operatorname{ex}(\varphi_t)$. The proof of Proposition 2 in [**3**] shows that there is a (unique) continuous linear map $L\colon \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$ such that near $t = 0$

$$\Phi_t(x) = x + tL(f)(x) + o(t) \tag{2.3}$$

whenever $x \in B^n$ and f is smooth.

THEOREM 2. *If $L\colon \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$ satisfies* (2.3), *then*

$$L = \frac{n}{2(n-1)} L_0.$$

PROOF. First we show that L is conformally natural. It suffices to verify (1.2) for smooth f. Let φ_t satisfy (2.2). If $g \in G_n$, an easy computation gives

$$g\varphi_t g^{-1}(u) = u + t(g \cdot f)(u) + o(t), \qquad u \in S^{n-1}.$$

Therefore

$$\operatorname{ex}(g\varphi_t g^{-1})(x) = x + tL(g \cdot f)(x) + o(t).$$

But (2.1) and (2.3) give

$$\operatorname{ex}(g\varphi_t g^{-1})(x) = g\Phi_t g^{-1}(x) = x + t(g \cdot L(f))(x) + o(t).$$

This proves the conformal naturality of L.

The proof of Theorem 2 can be completed by direct computation of $L(f)(0)$, using the definition of $\operatorname{ex}(\varphi)(0)$. However, it is more instructive to use Theorem 1 and (2.1). Let $\varphi_0\colon S^{n-1} \to S^{n-1}$ be the identity map and $\Phi_0 = \operatorname{ex}(\varphi_0)\colon \operatorname{Cl} B^n \to \operatorname{Cl} B^n$. Then (2.1) gives

$$g \circ \Phi_0 = \operatorname{ex}(g \circ \varphi_0) = \operatorname{ex}(\varphi_0 \circ g) = \Phi_0 \circ g$$

for all $g \in G_n$. Therefore Φ_0 is the identity map, so $\operatorname{ex}(g) = g$ for all $g \in G_n$ by (2.1).

In particular, let the G_n-invariant vector field f on $B^n \cup S^{n-1}$ generate the one-parameter group

$$g_t(x) = e^{tf}(x) = x + tf(x) + o(t), \qquad x \in \operatorname{Cl} B^n,$$

in G_n. Since $\operatorname{ex}(g_t) = g_t$, we see that $L(f) = f$. We apply this to

$$f_n(x) = (1 + |x|^2)e_n - 2(x \cdot e_n)x, \qquad x \in \operatorname{Cl} B^n, \tag{2.4}$$

and obtain $L(f_n)(0) = f_n(0) = e_n$. Now (1.3) gives

$$\begin{aligned} L_0(f_n)(0) &= \int_{S^{n-1}} f_n(u)\, d\omega(u) = 2\int_{S^{n-1}} [e_n - (u \cdot e_n)u]\, d\omega(u) \\ &= \left(2 - \frac{2}{n}\right) e_n = \frac{2(n-1)}{n} L(f_n)(0). \end{aligned}$$

Since L is a multiple of L_0, this proves Theorem 2. Q.E.D.

COROLLARY. *$L(f)$ defines a continuous extension of f to* $\operatorname{Cl} B^n$.

PROOF. In [**1**, **2**] Ahlfors proved that $(n/2(n-1))L_0(f)$ defines a continuous extension of f. Q.E.D.

We could not guarantee continuity on $\operatorname{Cl} B^n$ in advance because the methods of [**3**] do not show whether the $o(t)$ term in (2.3) is uniform in x.

3. A lemma from representation theory. The orthogonal group $O(n)$ is a subgroup of G_n, and formula (1.1) gives

$$(A \cdot f)(x) = Af(A^{-1}x) \tag{3.1}$$

if $A \in O(n)$, $f \in \mathcal{T}(S^{n-1})$ or $\mathcal{T}(B^n)$, and $x \in S^{n-1}$ or B^n, respectively. In §4 we shall derive Theorem 1 from

LEMMA 1. *If $T\colon \mathbf{R}^n \to \mathcal{T}(S^{n-1})$ is a linear map such that*

$$T(Ax) = A \cdot T(x), \qquad A \in O(n),\ x \in \mathbf{R}^n, \tag{3.2}$$

then there is a constant c in $\mathbf{R}$ such that

$$T(x)(u) = c[x - (x \cdot u)u], \qquad x \in \mathbf{R}^n,\ u \in S^{n-1}. \tag{3.3}$$

PROOF. This lemma is an immediate consequence of the Frobenius Reciprocity Theorem (see, for instance [**4**, Chapter 1]). To make this paper self-contained, we adapt the proof of that theorem in [**4**] to our particular situation. We identify $\mathbf{R}^{n-1}$ with the subspace $(e_n)^\perp$ of $\mathbf{R}^n$ and $O(n-1)$ with the subgroup $\{A \in O(n); Ae_n = e_n\}$ of $O(n)$.

Given the linear map $T\colon \mathbf{R}^n \to \mathcal{T}(S^{n-1})$ satisfying (3.2), we define a linear map $P\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ by $P(x) = T(x)(e_n)$. For any A in $O(n-1)$ and x in $\mathbf{R}^n$, (3.2) and (3.1) give

$$\begin{aligned} P(Ax) &= T(Ax)(e_n) = (A \cdot T(x))(e_n) \\ &= AT(x)(e_n) = AP(x). \end{aligned}$$

It follows easily that P is a multiple of the orthogonal projection of $\mathbf{R}^n$ onto $\mathbf{R}^{n-1}$. Thus

$$T(x)(e_n) = c[x - (x \cdot e_n)e_n]$$

for some constant c in $\mathbf{R}$. Finally, if $A \in O(n)$,

$$\begin{aligned} T(x)(Ae_n) &= AA^{-1}T(x)(Ae_n) = A(A^{-1} \cdot T(x))(e_n) \\ &= AT(A^{-1}x)(e_n) = cA[A^{-1}x - (A^{-1}x \cdot e_n)e_n] \\ &= c[x - (x \cdot Ae_n)Ae_n]. \end{aligned}$$

This proves (3.3). Q.E.D.

REMARK. The linear map T defined by (3.3) satisfies (3.2), so the space of linear maps T that satisfy (3.2) has dimension one. Equivalently (see [**4**, Chapter 1]), the standard representation of $O(n)$ on $\mathbf{R}^n$ occurs exactly once in the representation (3.1).

4. Proof of Theorem 1. Given the continuous linear map $L\colon\ \mathcal{T}(S^{n-1}) \to \mathcal{T}(B^n)$ satisfying (1.2), we define the continuous linear map $\lambda\colon\ \mathcal{T}(S^{n-1}) \to \mathbf{R}^n$ by

$$\lambda(f) = L(f)(0), \qquad f \in \mathcal{T}(S^{n-1}). \tag{4.1}$$

Formulas (4.1), (1.2), and (3.1) imply

$$\lambda(A \cdot f) = A\lambda(f), \qquad A \in O(n),\ f \in \mathcal{T}(S^{n-1}). \tag{4.2}$$

For $k \geq 2$, let V_k be the subspace of $\mathcal{T}(S^{n-1})$ consisting of the f whose component functions can be defined by polynomials of degree $\leq k$. The subspace V_k is $O(n)$-invariant, and so is the inner product

$$\langle f, h\rangle = \int_{S^{n-1}} (f(u) \cdot h(u))\, d\omega(u), \qquad f, h \in V_k. \tag{4.3}$$

LEMMA 2. *There is a constant c in $\mathbf{R}$ such that*

$$\lambda(f) = c\lambda_0(f), \qquad f \in \mathcal{T}(S^{n-1}). \tag{4.4}$$

Here $\lambda_0(f) = L_0(f)(0)$.

PROOF. We saw in §2 that

$$\lambda_0(f_n) = \frac{2(n-1)}{n} e_n \neq 0$$

if f_n in $\mathcal{T}(S^{n-1})$ is defined by (2.4). Therefore, the constant c in (4.4) must equal $\lambda(f_n)/\lambda_0(f_n)$. By the Stone-Weierstrass theorem, it suffices to prove

$$\lambda(f) = (\lambda(f_n)/\lambda_0(f_n))\lambda_0(f), \qquad f \in V_k, \tag{4.5}$$

for each $k \geq 2$. Observe that $f_n \in V_k$ (since $|x| = 1$ on S^{n-1}), so $\lambda_0\colon\ V_k \to \mathbf{R}^n$ is nontrivial. If $\lambda\colon\ V_k \to \mathbf{R}^n$ is trivial, then (4.5) is trivially satisfied because both sides of the equation are zero. If $\lambda\colon\ V_k \to \mathbf{R}^n$ is nontrivial, its kernel U_k is an $O(n)$-invariant subspace of V_k. The orthogonal complement $W_k = U_k^{\perp}$ is also $O(n)$-invariant, because of the invariance of the inner product (4.3).

Now $\lambda(W_k) = \lambda(V_k)$ is a nontrivial $O(n)$-invariant subspace of $\mathbf{R}^n$, so $\lambda(W_k) = \mathbf{R}^n$, and $\lambda\colon\ W_k \to \mathbf{R}^n$ is an isomorphism that satisfies (4.2). The inverse map

$$T = \lambda^{-1}\colon\ \mathbf{R}^n \to W_k \subset \mathcal{T}(S^{n-1})$$

satisfies (3.2), so it also satisfies (3.3), by Lemma 1. Thus $W_k = T(\mathbf{R}^n)$ is independent of λ, and so is $U_k = W_k^{\perp}$. We conclude that λ, like T, is unique up to multiplication by a constant, so λ is a multiple of λ_0 in V_k. Equation (4.5) follows at once. Q.E.D.

Theorem 1 follows immediately from (4.4) because λ determines L. Indeed, if $x \in B^n$, $g \in G_n$, and $g(x) = 0$, then (1.1), (1.2), (4.1), and (4.4) give

$$\begin{aligned} g'(x)L(f)(x) &= (g \cdot L(f))(0) = L(g \cdot f)(0) = \lambda(g \cdot f) \\ &= c\lambda_0(g \cdot f) = cg'(x)L_0(f)(x), \end{aligned}$$

so $L(f) = cL_0(f)$. Q.E.D.

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