THE GELFAND THEOREM AND ITS CONVERSE
FOR KÄHLER MANIFOLDS

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ABSTRACT. We characterize the locally Hermitian symmetric manifolds among the homogeneous Kähler manifolds \( M \) by each of the following properties:

(i) all \( A_0(M) \)-invariant differential operators on \( M \) commute (\( A_0(M) \) denotes the identity component of the group of all holomorphic isometries);
(ii) all geodesics are orbits of one-parameter groups of holomorphic isometries.

Recently, D’Atri, Dorfmeister, and Zhao Yan da [1] proved the following characterization of symmetric Siegel domains.

**THEOREM A.** Let \( D \) be a homogeneous Siegel domain and \( G \) the identity component of the automorphism group of \( D \). Then the algebra of \( G \)-invariant differential operators on \( D \) is commutative if and only if \( D \) is a symmetric domain.

The “if” part is an easy modification of the well-known Gelfand theorem (see [3]). It still holds in the following more general situation. Let \( M \) be a Hermitian symmetric space (thus a homogeneous Kähler manifold), and let \( A_0(M) \) denote the identity component of the group of all holomorphic isometries of \( M \). Then the algebra of \( A_0(M) \)-invariant differential operators on \( M \) is commutative.

In this report we prove a converse of the last statement and this gives an essential generalization of Theorem A.

**THEOREM 1.** Let \((M,g,J)\) be a homogeneous Kähler manifold. If all \( A_0(M) \)-invariant differential operators on \( M \) commute, then \( M \) is locally Hermitian symmetric.

**COROLLARY.** Let \( M \) be a simply connected homogeneous Kähler manifold. Then the algebra of \( A_0(M) \)-invariant differential operators on \( M \) is commutative if and only if \( M \) is Hermitian symmetric.

The proof is based on two lemmas.

**LEMMA 1** [7]. Let \((M,g)\) be a smooth \( n \)-dimensional Riemannian manifold. Let

\[
D = \sum_{i_1, \ldots, i_k = 1}^{n} T^{i_1 \ldots i_k} \nabla_{i_1 \ldots i_k}^k
\]

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be a differential operator whose coefficients are symmetric and which commutes with the Laplacian $\Delta = \sum_{i,j=1}^{n} g^{ij} \nabla_i^2$. Then the following tensor identity holds:

$$ (1) \quad \nabla_{i_1} T_{i_2 \ldots i_{k+1}} + \cdots + \nabla_{i_k+1} T_{i_1 \ldots i_k} = 0. $$

A short proof can be found in [4].

**Lemma 2.** A Kähler manifold $(M, g, J)$ is locally Hermitian symmetric if and only if

$$ (2) \quad (\nabla_X R)(X, JX, X, JX) = 0 $$

holds for each tangent vector $X$.

The identity (2) was first investigated by A. Gray in connection with the theory of 3-symmetric spaces, and the proof of Lemma 2 can be obtained by combining Theorem 4.6 and Corollary 4.4 from [2]. A full direct proof (elementary, but rather long) can be found in two parts in [6] and [8]. Although this lemma proved to be useful in many topics of Kählerian geometry, a really short and simple proof is not yet available.

**Proof of Theorem 1.** Define a 4-valent tensor field $T$ on $M$ by putting

$$ T^{ijkl} = T_{ijkl} = R(e_i, Je_j, e_k, Je_l) + R(e_k, Je_j, e_i, Je_l) + R(e_l, Je_j, e_k, Je_i) + R(e_i, Je_j, e_l, Je_k) $$

with respect to any orthonormal frame (where $R$ is the curvature tensor). Since $M$ is Kählerian, $T^{ijkl}$ is symmetric. Further, $T$ is $A_0(M)$-invariant and hence the differential operator

$$ D = \sum_{i,j,k,l=1}^{n} T^{ijkl} \nabla^{4}_{ijkl} $$

is also $A_0(M)$-invariant. Then $D$ must commute with $\Delta$ and Lemma 1 gives $(\nabla_X T)(X, X, X, X) = 0$, i.e. $(\nabla_X R)(X, JX, X, JX) = 0$ for each tangent vector $X$. Now we use only Lemma 2 to obtain the result.

**Note A.** The converse of the Gelfand theorem is not true in the real case. For instance, there are many naturally reductive homogeneous Riemannian manifolds for which all $I_0(M)$-invariant differential operators commute and which are not locally symmetric. Yet, the following weaker converse still holds (see [4]): Let $(M, g)$ be a homogeneous Riemannian manifold and $I_0(M)$ the identity component of the full isometry group of $M$. If all $I_0(M)$-invariant differential operators on $M$ commute, then the local geodesic symmetries of $(M, g)$ are volume-preserving.

**Note B.** The following result is related to our Theorem 1.

**Theorem 2.** Let $(M, g, J)$ be a homogeneous Kähler manifold all of whose geodesics are orbits of one-parameter groups of holomorphic isometries. Then $(M, g, J)$ is locally Hermitian symmetric.

**Proof.** Along any fixed geodesic $\gamma$, the function $R(\gamma', J\gamma', \gamma', J\gamma')$ is constant. Hence $\gamma'[R(\gamma', J\gamma', \gamma', J\gamma')] = 0$, and because $M$ is Kählerian, we get the condition of Lemma 2.

Again, a real analogue holds in the weaker form (see [5]): Let $(M, g)$ be a homogeneous Riemannian manifold all of whose geodesics are orbits of one-parameter groups of isometries. Then the local geodesic symmetries of $(M, g)$ are volume-preserving.
REFERENCES


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