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SOME AMALGAM STRUCTURES FOR BIANCHI GROUPS

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ABSTRACT. Splittings as an amalgamated sum, where one of the factors is the projective elementary matrices, are displayed for all the Bianchi groups except for the Euclidian ones. Also splittings as an HNN group, where the base is $\mathrm{PSL}_2(Z)$, are given for all the Bianchi groups except $\mathrm{PSL}_2(O_1)$ and $\mathrm{PSL}_2(O_3)$.

0. Introduction. This paper grew out of a seminar on the Bianchi groups. In addition to the authors, Seymour Bachmuth and Morris Newman were active participants. Without their knowledge and enthusiasm this paper would not exist.

Our knowledge of the modular group stands as the basic example upon which the study of infinite groups is modeled. Since $\mathrm{PSL}_2(\mathbf{Z})$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ we can view the modular group as a group of isometries of the hyperbolic plane H^2 acting in a properly discontinuous fashion. The interplay between the number theory of the rational integers and the geometry of $\mathrm{PSL}_2(\mathbf{Z})\backslash H^2$ has led to the beautiful theory of the modular group. Let d be a positive square free integer, and let O_d denote the ring of integers in the field $\mathbf{Q}(\sqrt{-d})$. After the rational integers, the quadratic imaginary number rings O_d are the most completely understood examples in number theory. The groups $\Gamma_d = \mathrm{PSL}_2(O_d)$ are known as the Bianchi groups and have been extensively studied. Since Γ_d is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ we can view the Bianchi groups as isometries of hyperbolic space H^3 acting in a properly discontinuous fashion. Hence one would expect that the geometry of $\Gamma_d\backslash H^3$ and the number theory of O_d could be used to derive information about Γ_d in much the same way that the modular group has been studied. In fact study of the geometry of Γ_d goes back to Bianchi [**B1**], although the work of Swan [**Sw**] is the fundamental reference for modern studies. In [**R**], Riley describes a computer program based on Swan's work that computes the Ford domains of the Bianchi groups. Also Hatcher [**H**] has drawn the orbifolds of $\mathrm{PGL}_2(O_d)$ for all $d < 100$.

The question that has driven research in this direction in recent years is the cuspidal cohomology question. Simply stated the question is: For which d is the first rational homology of $\Gamma_d\backslash H^3$ carried by the cusps? This question has been answered in the papers [**B**, **GS**, **Ro**, **V** and **Z**]. Specifically the answer is when $d \in \{1,2,3,5,6,7,11,15,19,23,31,39,47,71\}$.

In this paper we use results appearing in [**Sw**] to show the existence of a family of nontrivial splittings of Γ_d. Throughout this paper $\{1,\omega\}$ will denote the standard module basis for O_d. The projective elementary group E_d is the subgroup of Γ_d generated by $a = -1/z$, $t = z+1$ and $\mu = z+\omega$. If O_d is not Euclidean then we

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show that $\Gamma_d \cong E_d *_{F_d} G_d$ where F_d is a subgroup of E_d that is canonically defined and G_d is a group whose structure depends on d. This splitting comes from an incompressible surface in $\Gamma_d\backslash H^3$, which in contrast to the splittings in [**H**, **Z**] is not totally geodesic. The basic fact that leads to the preceding result is that for $d \neq 1$ or 3, Γ_d is an HNN group amalgamated over $\mathrm{PSL}_2(\mathbf{Z})$. We also find splittings of Γ_2, Γ_7, and Γ_{11} as amalgamated sums. It is known that Γ_1 splits as an amalgamated sum over $\mathrm{PSL}_2(\mathbf{Z})$ and that Γ_3 does not admit splittings as an amalgamated sum [**S**].

The following theorem appears in [**CS**].

THEOREM. *Let M be a compact, connected, orientable 3-manifold having a nonempty family of tori as boundary. Suppose that $H_1(\partial M, \mathbf{Q}) \to H_1(M, \mathbf{Q})$ is surjective and that M is not homeomorphic to $D^2 \times S^1$ or $S^1 \times S^1 \times I$. Then M contains a separating, (properly embedded) incompressible surface which has nonempty boundary and is not boundary parallel.* □

The existence of a separating incompressible surface that is not boundary parallel implies that $\pi_1(M)$ splits as an amalgamated sum. If Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ and $\Gamma\backslash H^3$ has finite volume, then $\Gamma\backslash H^3$ is homeomorphic to the interior of a compact three-manifold. The rational homology of $\Gamma\backslash H^3$ is naturally isomorphic to the rational homology of Γ. Hence it makes sense to refer to the peripheral homology of Γ (homology coming from the boundary). From the solution of the cuspidal cohomology problem we know that only 14 Bianchi groups have the property that their rational first homology is generated by their peripheral homology. Hence only for $d \in \{2, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$ does Culler and Shalen's proof adapt to show that Γ_d is an amalgamated sum. It was out of a desire to sharpen their results that we took about to understand the other Bianchi groups.

1. Basic concepts. Realize H^3 as $\{(z, \varsigma) \mid z, \varsigma \in \mathbf{C}, \varsigma > 0\}$ using the standard upperhalf space metric. The group of orientation preserving isometries of H^3 is isomorphic to $\mathrm{PSL}_2(\mathbf{C})$ where the action is given as follows. Suppose $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{C})$ represents an element of $\mathrm{PSL}_2(\mathbf{C})$. Then $\sigma(z, \varsigma) = (z', \varsigma')$, where

$$z' \frac{(\bar{d} - \bar{c}\bar{z})(az - b) - \varsigma^2 \bar{c} a}{|cz - d|^2 + \varsigma^2 |c|^2} \quad \text{and} \quad \varsigma' = \frac{\varsigma}{|cz - d|^2 + \varsigma^2 |c|^2}.$$

If Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ then the Ford domain B of Γ is the region of H^3 where all $\sigma \in \Gamma$ have Jacobian determinant less than or equal to one. If Φ is the subgroup of Γ that fixes infinity and if D is a fundamental domain for Φ then $B \cap D$ is a fundamental domain for Γ. In [**Sw**], a method is given for computing the Ford domains of the Bianchi groups. Specifically

$$B = \{(z, \varsigma \mid |\mu z - \lambda|^2 + \varsigma^2 |\mu|^2 \geq 1 \ \forall \lambda, \mu \in O_d, (\lambda, \mu) = O_d\},$$

where $(\lambda, \mu) = O_d$ means that λ and μ generate O_d. Furthermore [**Sw**] it is shown that over the line $\operatorname{im} z = 0$ the lower boundary of B is made up of the spheres of radius one centered at the integers. More precisely $B \cap \{\operatorname{im} z = 0\} = \{(x, \varsigma) \in \mathbf{R} \times \mathbf{R}_{>0} \colon \forall n \in \mathbf{Z}, |x - n|^2 + \varsigma^2 \geq 1\}$.

If Γ acts properly discontinuously on H^2 or H^3 then $\Gamma\backslash H^n$ inherits the structure of an orbifold [**Th**]. Although the orbifold concept clarifies the nature of our studies, the geometric and topological preliminaries at this state of development are rather

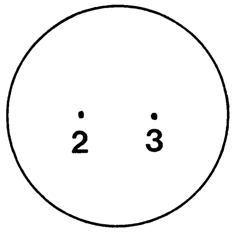

FIGURE 1

forboding. Hence we choose to follow Zieschang [**Z1**] and state our results inside the framework of factor manifolds.

DEFINITION 1.1. Let M be a manifold, with or without boundary, of dimension $n = 2$ or 3. Let S be a codimension 2 CW-complex that is properly embedded in M. Assume also that each $n-2$ cell of S has a positive integer assigned to it. We call S the *branch set* of M. Call the number h_i assigned to the cell e_i of S the *index* of e_i. We call the pair (M, S) a *factor manifold.*

EXAMPLE 1.2. If Γ is a Fuchsian group then $F = \Gamma\backslash H^2$ is a surface. The points on F that correspond to points in H^2 having nontrivial stabilizers will make up the branch set T. To each $q \in T$ assign the order of the stabilizer of a lift of q to H^2. In Figure 1 we have pictured the factor manifold corresponding to $\mathrm{PSL}_2(\mathbf{Z})\backslash H^2$.

EXAMPLE 1.3. If Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ then $M = \Gamma\backslash H^3$ is a three-manifold. We let the branch set S be the subset of M corresponding to points in H^3 having nontrivial stabilizers. We can give S a cell structure by letting the vertices be the nonmanifold points of S and the 1-cells be the arcs running between vertices. If some component of S is a circle then just choose an arbitrary point on that component to be a vertex. Once again the index of a 1-cell e is the order of the stabilizer of one of its lifts to H^3. In Figure 2 we have pictured the factor manifolds corresponding to $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_{11}$.

Suppose that Γ is a discrete group of orientation preserving isometries of H^2 or H^3. We would like to use the Seifert-Van Kampen theorem in our study of Γ; hence we construct a complex that has Γ as its fundamental group. At this point in our discussion we need to deal with surfaces and three-manifolds separately. First we will deal with factor surfaces, then with factor three-manifolds.

DEFINITION 1.4. Let (F, T) be a factor surface. Replace a regular neighborhood $N(e_i)$ of each 0-cell e_i in T by a disk D_i so that the gluing map $H_i \colon \partial D_i \to \partial N(e_i)$ has degree h_i, the index of e_i. Denote the complex we have constructed by $\tilde{F}$.

In [**Z1**] it is remarked that if (F, T) is the factor manifold corresponding to a discrete group of isometries Γ of H^2 then $\pi_1(\tilde{F}) \cong \Gamma$.

DEFINITION 1.5. Let (M, S) be a factor three-manifold. Let $N(S)$ be a regular neighborhood of S in M. We can decompose $N(S)$ into balls and beams. The balls are regular neighborhoods of the 0-cells of S in M. The beams are regular neighborhoods in the complement of the balls of the 1-cells of S (see Figure 3). This decomposition of $N(S)$ induces a decomposition of $\partial N(S)$ into annuli that are the frontiers of beams in $\hat{M} = \mathrm{Cl}(M - N(S))$ and planar surfaces which are the frontiers of balls in $\hat{M}$. Denote by A_i the annulus corresponding to the 1-cell e_i.

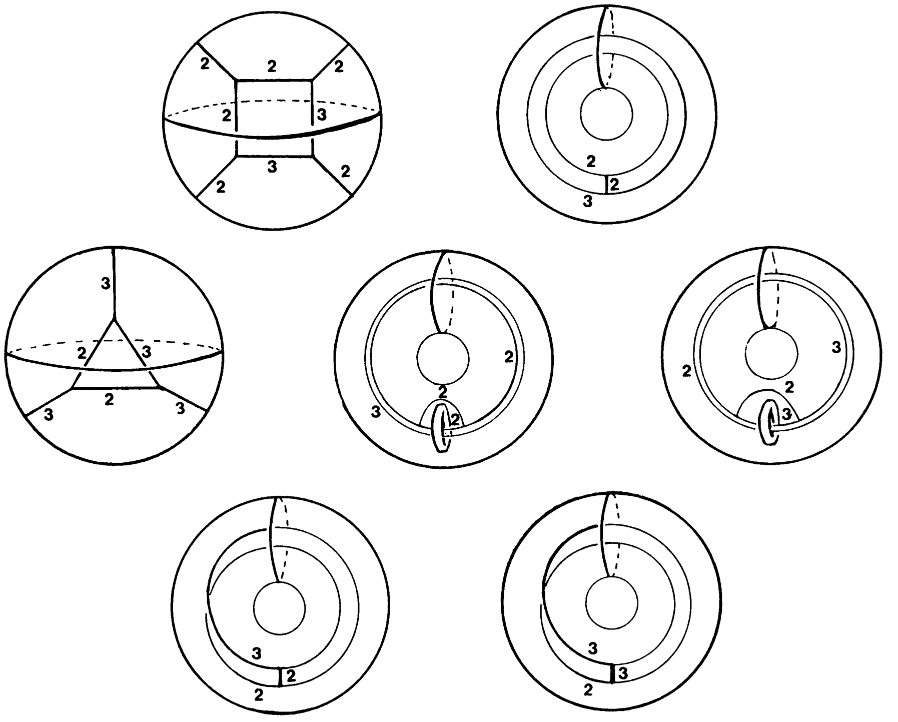

FIGURE 2. From left to right, top to bottom, the factor manifolds corresponding to $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_{11}$.

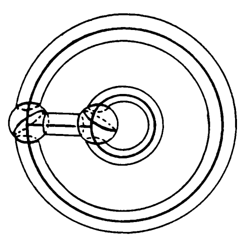

FIGURE 3

Let $H_i \colon \partial D^2 \times I \to A_i$ be a map of degree h_i. Let $\tilde{M}$ be the complex obtained by glueing a copy of $D^2 \times I$ onto $\hat{M}$ for each A_i using the maps H_i as glueing maps.

PROPOSITION 1.6. *Let Γ be a discrete subgroup of isometries of H^3. Let (M, S) be the factor manifold corresponding to Γ. Then $\Gamma \cong \pi_1(\tilde{M})$.*

PROOF. Let $q \colon H^3 \to M$ be the quotient map. The restriction of q to the complement $\hat{H}$ of $q^{-1}(\mathring{N}(S))$ is a regular covering of $\hat{M}$. Hence $\Gamma \approx \pi_1(\hat{M})/q_{\#}\pi_1(\hat{H})$. Let x_i be a meridian of the 1-cell e_i. To construct such a loop, run an arc a_i that misses $N(S)$ from the basepoint out to the boundary of a small disk D_i that intersects S transversely in a single point lying in the interior of e_i, around the boundary of D_i and back along a_i. The image of $\pi_1(\hat{H})$ in $\pi_1(\hat{M})$ is normally generated by $x_i^{h_i}$ where i ranges over the indices of 1-cells of S. By construction $\pi_1(M)$ is isomorphic to Γ. □

DEFINITION 1.7. Let F be a surface that is embedded in the factor three-manifold (M, S). If F intersects S transversely and misses the 0-cells of S, then we say that F is embedded nicely with respect to S. We obtain the induced factor manifold structure on F by letting the branch set T be the set of points $F \cap S$ equipped with the ramification indices of the corresponding 1-cells of S. Notice that if care is taken in the construction of $\tilde{F}$ and $\tilde{M}$ then $\tilde{F}$ can be viewed as a subcomplex of $\tilde{M}$. We say that F is incompressible in (M, S) if the inclusion map of $\tilde{F}$ into $\tilde{M}$ induces a monomorphism of $\pi_1(\tilde{F})$ into $\pi_1(\tilde{M})$.

Suppose that (M, S) is the factor manifold corresponding to $\Gamma \backslash H^3$ for some discrete $\Gamma \subseteq \mathrm{PSL}_2(\mathbf{C})$. Let F be a surface that intersects S nicely. Let Φ be a component of the inverse image of F in H^3. Then the stabilizer of Φ in Γ is conjugate to the image of $\pi_1(\tilde{F})$ in $\pi_1(\tilde{M}) \cong \Gamma$.

PROPOSITION 1.8. *Let F be a nicely embedded two-sided surface in (M, S). If F is incompressible then $\pi_1(\tilde{M})$ is either an amalgamated sum over $\pi_1(\tilde{F})$ or an HNN group over $\pi_1(\tilde{F})$, depending on whether F is separating or not.*

PROOF. Proposition 1.8 is a direct application of the Seifert-Van Kampen Theorem. □

2. Some splittings of the Bianchi groups. In this section we show that all of the Bianchi groups except Γ_3 split as amalgamated sums. Solitar [**S**] has shown that Γ_3 does not admit a splitting as an amalgamated sum. In Theorem 2.1 we summarize our findings for Γ_d when O_d is Euclidean. The splitting given for Γ_1 is common knowledge, our reference is [**Fl**]. The splittings for $\Gamma_2, \Gamma_7, \Gamma_{11}$ were indicated by incompressible annuli, although the actual presentations were derived from existing presentations. In Theorem 2.2 we show that for $d \neq 1, 3$, Γ_d is an HNN group having amalgamated subgroup isomorphic to $\mathrm{PSL}_2(\mathbf{Z})$. This is achieved by showing that the factor manifold for $\mathrm{PSL}_2(\mathbf{Z})$ embeds as a nonseparating incompressible surface in $\Gamma_d \backslash H^3$. In Theorem 2.4 we show that when O_d is non-Euclidean, $\Gamma \cong E_d *_{F_d} G_d$ where E_d is the projective elementary matrices and F_d is a particular subgroup corresponding to a nicely embedded separating incompressible sphere in $\Gamma_d \backslash H^3$ having four branch points, two of index 2 and two of index 3. When O_d is Euclidean $\Gamma_d \cong E_d$, so that this surface cannot yield a nontrivial splitting. Although arrived at constructively our original hope was to use valuation theoretic methods as in [**CS**] to derive the existence of nontrivial splittings as amalgamated sums. It is interesting to note the similarity of the splitting $\Gamma_d \cong E_d *_{F_d} G_d$ and the splitting of Nagao's theorem, $\mathrm{GL}_2(K[f]) \cong \mathrm{GL}_2(k) *_{B(k)} B(k[f])$.

In the following D_2 will denote the Klein four-group, S_3 the symmetric group on three letters, and A_4 the alternating group.

THEOREM 2.1. *The Bianchi groups $\Gamma_1, \Gamma_2, \Gamma_7$, and Γ_{11} are nontrivial free products with amalgamation. In particular*

(i) $\Gamma_1 = G_1 *_H G_2$ *where*
  (a) $G_1 = S_3 *_{Z_3} A_4$,
  (b) $G_2 = S_3 *_{Z_2} D_2$ *and*
  (c) $H = \mathrm{PSL}_2(\mathbf{Z})$;

(ii) $\Gamma_2 = G_1 *_H G_2$ *where*
(a) G_1 *is an HNN group whose base is* D_2 *with two elements of order* 2 *amalgamated,*
(b) G_2 *is an HNN group whose base is* A_4 *with two three-cycles amalgamated,*
(c) $H = Z * Z_2$;
(iii) $\Gamma_7 = G_1 *_H G_2$ *where*
(a) $G_1 = Z * Z_2$,
(b) G_2 *is an HNN group with base* $K = S_3 *_{Z_3} S_3$ *with two three-cycles, one from each* S_3, *amalgamated,*
(c) $H = Z * Z_2 * Z_2$;
(iv) $\Gamma_{11} = G_1 *_H G_2$ *where*
(a) $G_1 = Z * Z_3$,
(b) G_2 *is an HNN group with base* $A_4 *_{Z_3} A_4$ *and two three-cycles, one from each* A_4, *amalgamated,*
(c) $H = Z * Z_3 * Z_3$.

PROOF. The decomposition of Γ_1 appears in [**Fi1**]. Because of the similarity of the computations we will describe how we arrived at the presentation for Γ_2. We will then give the cases Γ_7 and Γ_{11} a light brushing over.

In [**Fi2**] the following presentation of Γ_2 is given, where $\mu = z + \omega$, $a = -1/z$ and $t = z + 1$:

$$\Gamma_2 = \langle \mu, a, t; a^2 = (at)^3 = (\mu a \bar{\mu} a)^2 = [t, \mu] = 1 \rangle.$$

Letting $s = at$, $v = \bar{\mu} s \mu$ and $m = \bar{\mu} a \mu$ and applying the appropriate Tietze transformations we get $\Gamma_2 = \langle \mu, a, s, v, m; a^2 = s^3 = m^2 = (am)^2 = (s\bar{v})^2 = 1, \bar{\mu} a = m, am = s\bar{v}, \bar{\mu} s = v \rangle$. From this presentation we see that Γ_2 is the amalgamated sum of $G_1 = \langle a, m, u; a^2 = m^2 = (am)^2 = 1, \bar{\mu} a \mu = m \rangle$ and $G_2 = \langle s, v, \mu; s^3 = v^3 (s\bar{v})^2 = 1, \bar{u} s = v \rangle$ with identifications $\mu = \mu$ and $s\bar{v} = am$. Clearly G_1 is an HNN extension of the Klein four-group with two elements of order two identified, and G_2 is an HNN extension of A_4 with two three-cycles identified. To complete the proof it must be shown that $\langle am, \mu \rangle$ is isomorphic to $\langle s\bar{v}, \mu \rangle$ under the given homomorphism and that these two groups are isomorphic to $Z * Z_2$. Suppose that an HNN group is given with base group G and free part μ and amalgamated subgroups H, K; that is $\bar{\mu} H \mu = K$. It is a consequence of of the normal form theorem for HNN groups that if L is a subgroup of the base group that has trivial intersection with the amalgamated subgroups H and K then the group $\langle L, \mu \rangle$ is the free product of L and $\langle \mu \rangle$. This applies in both the situations above, hence the proof is complete.

In [**Fi2**] the following presentation for Γ_7 is given:

$$\Gamma_7 = \langle a, v, s, m, w; a^2 = v^3 = (av)^2 = 1, av = ms^2, \bar{w} a w = m, \bar{w} s w = v \rangle,$$

where $a = -1/z, m = (-\omega z + 1 - \omega)/(z + \omega)$, $v = ((1 - \omega)z + 1)/(z + \omega)$, $s = -1/(z + 1)$, and $w = -1/(z + \omega)$. Letting $t = w^2$ and $x = av$ and making the appropriate Tietze transformations we get $\Gamma_7 = \langle a, v, w, t, x; a^2 = v^3 = (av)^2 x^2 = 1,\ wxw = at\bar{v},\ t = w^2,\ x = av \rangle$. From this we see that Γ_7 is the amalgamated product of $G_1 = \langle xw; x^2 = 1 \rangle$ and $G_2 = \langle a, v, t; a^2 = v^3 = (av)^2 (at\bar{v}\bar{t})^2 = 1 \rangle$ with identifications $x = av$, $w^2 = t$ and $wxw = at\bar{v}$. The rest is similar to the above computations.

Interpretating $\{1, \omega\}$ to be the standard module basis for Γ_{11} instead of Γ_7 we have that $\Gamma_{11} = \langle a, v, s, m; a^2 = s^3 = (av)^3 = 1,\ av = sm,\ \bar{w}aw = m,\ \bar{w}sw = v\rangle$ where a, v, s, m, and w have the formulas given above. The same substitutions as we made in the Γ_7 case yield the appropriate presentation. □

THEOREM 2.2. *If* $d \neq 1, 3$ *then* Γ_d *is an HNN group with amalgamated subgroup* $\mathrm{PSL}_2(\mathbf{Z})$.

PROOF. Let H^2 be $\operatorname{im} z = 0$ in H^3. Clearly the stabilizer of H^2 in Γ_d contains $\mathrm{PSL}_2(\mathbf{Z})$. From [**Sw**, Lemma 8.2] we have that $B \cap H^2$ coincides with the Ford domain of $\mathrm{PSL}_2(\mathbf{Z})$ acting on H_2. Since a fundamental domain for the stabilizer of H^2 is the intersection of its Ford domain (in H^2) and a fundamental domain for its subgroup of elements fixing infinity, we have that the stabilizer of H^2 is generated by $\mathrm{PSL}_2(\mathbf{Z})$ and some transformations that fix infinity. Except for Γ_1 and Γ_3 this means that the additional transformations are of the form $z + \gamma$ where $\gamma \in O_d$. If such a map stabilizes H^2, γ must be in $\mathbf{Z}$. Thus we have that for $d \neq 1, 3$ the stabilizer of H^2 is $\mathrm{PSL}_2(\mathbf{Z})$. To show that the image of H^2 in $\Gamma_d \backslash H^3$ is embedded we must show that if $\sigma(H^2) \cap H^2 \neq \varnothing$ for some $\sigma \in \Gamma_d$ then $\sigma(H^2) = H^2$. Suppose that $\sigma(H^2) \cap H^2 \neq \varnothing$; we may precompose and postcompose σ by elements of $\mathrm{PSL}_2(\mathbf{Z})$ to obtain τ so that τ carries a point in the Ford domain of $\mathrm{PSL}_2(\mathbf{Z})$, in H^2, to another point in the Ford domain of $\mathrm{PSL}_2(\mathbf{Z})$. Since the Ford domain for $\mathrm{PSL}_2(\mathbf{Z})$ lies inside the Ford domain for Γ_d we have that τ fixes infinity. As above $\tau(z) = z + \gamma$. Since $\tau(H^2) \cap H^2 \neq 0$ we have that $\gamma \in Z$. Thus $\tau(H^2) \cap H^2 = H^2$ and consequently $\sigma(H^2) = H^2$. Hence the image of H^2 in $\Gamma_d \backslash H^3$ is an embedded two-sided surface Σ. From the structure of $H^2 \cap B$ we infer that Σ is a proper open disk intersecting the branch set nicely in two points, one of index 2 and one of index 3. Let $\tilde{\Sigma} \subset \widetilde{\Gamma_d \backslash H^3}$ be the associated complexes as in §1. We get the presentation $\pi_1(\tilde{\Sigma}) = \langle x_1, x_2; x_1^2 = x_2^3 = 1\rangle$. From the considerations above we see that we can choose basepoints so that $\pi_1(\tilde{\Sigma}) \to \pi_1(\widetilde{\Gamma_d \backslash H^3}) \to \Gamma_d$ induces a surjection $\pi_1(\tilde{\Sigma}) \twoheadrightarrow \mathrm{PSL}_2(\mathbf{Z})$. Since all nonelementary Fuchsian groups are Hopfian, this implies that the inclusion map $\pi_1(\tilde{\Sigma}) \to \pi_1(\widetilde{\Gamma_d \backslash H^3})$ is a monomorphism. Therefore Σ is incompressible. Let α be a straight line from (0, 2) to $(\omega, 2)$ in H^3. The arc α has been chosen so that it lies in the Ford domain of Γ_d. Its endpoints lie in H^2 and the translate of H^2 by $z + \omega$, so that $z + \omega$ maps one endpoint to the other, and its interior is disjoint from the translates of H^2. This allows us to conclude that the image of α in $\Gamma_d \backslash H^3$ is a simple closed curve intersecting Σ in a single point of transverse intersection. Hence Σ is nonseparating. The result now follows from Proposition 1.8. □

Our next goal is to show that when O_d is not Euclidean then Γ_d splits as an amalgamated sum. To do this we need the following lemma.

LEMMA 2.3. *Suppose* $d \neq 1, 2, 3, 7, 11$, *let*

$$x_1 = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x_2 = \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad x_3 = \pm\begin{pmatrix} \omega & -\omega^2 - 1 \\ 1 & -1 \end{pmatrix},$$

$$x_4 = \pm\begin{pmatrix} -\omega & \omega^2 - \omega + 1 \\ -1 & -1 + \omega \end{pmatrix}.$$

The subgroup F_d of $\mathrm{PSL}_2(O_d)$ *generated by* x_1, x_2, x_3, x_4 *has the presentation*

$$\langle x_1, x_2, x_3, x_4; x_1^2 = x_2^3 = x_3^2 = x_4^3 = x_1x_2x_3x_4 = 1\rangle.$$

PROOF. Let $a = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $t = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\mu = \left(\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right)$. Then in terms of these the above four transformations are given by $x_1 = a$, $x_2 = at$, $x_3 = \mu a \mu^{-1}$, $x_4 = \mu a t \mu^{-1}$. We note that x_1, x_2 generate the classical modular group $\mathrm{PSL}_2(\mathbf{Z})$.

From a result of Fine [**Fi1**], for any $d \neq 1, 2, 3, 7, 11$ the elements a, t, μ generate the projective elementary group $PE_2(O_d) = E_d$ and a complete presentation for E_d is given by

$$E_d = \langle a, t, \mu; s^2 = (at)^3 = [t, \mu] = 1\rangle.$$

Furthermore this presentation is independent of d.

In E_d, consider the normal closure of the modular group. We can denote this by $N(a,t)$. Then $E_d/N(a,t) \cong \langle \mu \rangle$. Therefore $\{\mu^n\}$, $n \in \mathbf{Z}$, constitutes a complete set of coset representatives for $N(a,t)$ in E_d. Using these and applying the Reidemeister-Schreier rewriting process we have that $N(a,t)$ is generated by $a_i = \mu^i a \mu^{-i}$ and $t_i = \mu^i t \mu^{-i}$ where i ranges over $\mathbf{Z}$, with the relations $a_i^2 = (a_i t_i)^3 = 1$ and $t_i = t_{i+1}$. Let $G_i = \langle a_i, t_i; a_i^2 = (a_i t_i)^3 = 1\rangle$. Then $N(a,t)$ is an infinite tree product with the G_i's as vertices and amalgamations, $t_i = t_{i+1}$.

In a tree product a presentation for a connected subtree is the obvious one, so for the subgroup generated by G_0 and G_1,

$$\langle G_0, G_1\rangle = \langle a_0, a_1, t_0, t_1; a_0^2 = a_1^2 = (a_0t_0)^3 = (a_1t_1)^3 = 1, t_0 = t_1\rangle.$$

Noting that $a_0 = a = x_1$, $a_1 = \mu a \mu^{-1} x_2$, $a_0 t_0 = x_3$ and $a_1 t_1 = x_4$ we have that the subgroup of Γ_d generated by x_1, x_2, x_3 and x_4 has presentation $\langle x_1, x_2, x_3, x_4; x_1^2 = x_3^3 = x_2^2 = x_4^3 = x_1x_2x_3x_4 = 1\rangle$. □

THEOREM 2.4. *When $d \neq 1, 2, 3, 7, 11$, Γ_d admits a nontrivial splitting as $E_d *_{F_d} G_d$ where E_d and F_d are as above.*

PROOF. Let Σ denote the image of H^2 in $\Gamma_d \backslash H^3$. We can truncate the cusps of $\Gamma_d \backslash H^3$ to obtain a compact manifold M_d having tori as boundary so that $\Sigma \cap M_d$ is a proper disk intersecting the branch set nicely in two points. Let T_∞ denote the component of ∂M_d having nonempty intersection with Σ. Let $N(\Sigma \cup T_\infty)$ be a small regular neighborhood and let F be the frontier of $N(\Sigma \cup T_\infty)$. Notice that F is a separating sphere in $\Gamma_d \backslash H^3$ that intersects the branch set nicely in four points, two of index 2 and two of index 3. Let $\tilde{F}$ and $\widetilde{\Gamma_d \backslash H^3}$ be the associated complexes. From the construction of $\tilde{F}$ we see that $\pi_1(\tilde{F}) \cong \langle x_1, x_2, x_3, x_4; x_1^2 = x_2^3 = x_3^2 = x_4^3 = 1, x_1x_2x_3x_4 = 1\rangle$. The group corresponding to $N(\Sigma \cup T_\infty)$ is E_d and the group corresponding to its frontier is F_d. This can be seen by considering the inverse image of $N(\Sigma \cup T_\infty)$ in H^3. From Lemma 2.4 and the fact that nonelementary Fuchsian groups are Hopfian we conclude that $\pi_1(\tilde{F}) \to \pi_1(\widetilde{\Gamma_d \backslash H^3})$ is injective. By Proposition 1.8, $\Gamma_d \cong E_d *_{F_d} G_d$. Since O_d is not Euclidean, E_d is a proper subgroup of Γ_d [**Co**]; hence F_d is a proper subgroup of G_d and the splitting is nontrivial. □

BIBLIOGRAPHY

[**B1**] L. Bianchi, *Sui gruppi de sostituzioni lineari con coefficienti appartenenti a corpi quadratici imaginari*, Math. Ann. **40** (1897), 332–412.

[**B**] M. Baker, *Ramified primes and the homology of the Bianchi groups*, Inst. Hautes Études Sci., preprint, 1982.

[**Co**] P. Cohn, *On the structure of* GL_2 *of a ring*, Inst. Hautes Études Sci. Publ. Math. **30** (1966), 5–53.

[**CS**] M. Culler and P. Shalen, *Bounded separating surfaces in knot manifolds*, Invent. Math. **75** (1984), 537–545.

[**Fi1**] B. Fine, *The structure of* $\mathrm{PSL}_2(R)$, Ann. of Math. Studies, no. 79, Princeton Univ. Press, Princeton, N. J., 1974, pp. 145–170.

[**Fi2**] ——, *The HNN and generalized free product structure of certain linear groups*, Bull. Amer. Math. Soc. **81** (1975), 413–416.

[**Fl**] D. Flöge, *Zur Struktur der* PSL_2 *über einegen imaginär-quadratischen Zahlringen*, Math. Z. **183** (1983), 255–279.

[**GS**] F. Grünewald and J. Schwermer, *Arithmetic quotients of hyperbolic* 3*-space, cusp forms and link complements*, Duke Math J. **48** (1981), 183–200.

[**H**] A. Hatcher, *Bianchi orbifolds of small discriminant, an informal report*, Preprint.

[**R**] R. Riley, *Applications of a computer implementation of Poincaré's theorem on fundamental polyhedra*, Math. Comp. **40** (1983), 607–632.

[**Ro**] J. Rohlfs, *On the cuspidal cohomology of the Bianchi modular group*, Math. Z. **188** (1985), 253–270.

[**S**] D. Solitar, private correspondence.

[**Sw**] R. Swan, *Generators and relations for certain special linear groups*, Adv. in Math. **6** (1871), 1–78.

[**Th**] W. Thurston, *The geometry and topology of* 3*-manifolds*, mimeographed notes.

[**V**] K. Voightmann, *Rational homology of Bianchi groups*, Math. Ann. (to appear).

[**Z**] R. Zimmert, *Zur* SL_2 *der ganzen Zahlen eines imaginär-quadratischen Zahlkörpers*, Invent. Math. **19** (1973), 73–81.

[**Z1**] H. Zieschang, *Finite groups of mapping classes of surfaces*, Lecture Notes in Math., vol. 875, Springer-Verlag, Berlin and New York, 1981.

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