PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

PATH-LIFTING FOR GROTHENDIECK TOPOSES

IEKE MOERDIJK

(Communicated by Donald Passman)

ABSTRACT. A general path-lifting theorem, which fails in the context of topological spaces, is shown to hold for toposes, for locales (a slight generalization of topological spaces), and hence for complete separable metric spaces. This result generalizes the known fact that any connected locally connected topos (respectively complete separable metric space) is path-connected.

In **[MW]** we proved that every connected locally connected topos $\mathcal{E}$ is path-connected, in the sense that the canonical map

$$\mathcal{E}^I \to \mathcal{E} \times_S \mathcal{E} \tag{1}$$

given by the inclusion of the endpoints $\{0,1\} \subset I$ is a surjection (actually, we showed it to be an open surjection). Here I, the unit interval, is identified with the topos of sheaves on I, and $\mathcal{E}^I$ is the "path-space" of $\mathcal{E}$, i.e. there is an equivalence

$$\mathrm{Hom}_S(\mathcal{F}, \mathcal{E}^I) \simeq \mathrm{Hom}_S(\mathcal{F} \times_S I, \mathcal{E})$$

of categories of geometric morphisms over the base topos S, which is natural in $\mathcal{F}$.

The aim of this note is to point out that a much stronger result can in fact be proved. Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a connected locally connected map of toposes, i.e. $\mathcal{F}$ is connected and locally connected as an $\mathcal{E}$-topos (intuitively, the fibers of f are connected and locally connected). We will prove that for every path α in $\mathcal{E}^I$ and points y_0, y_1 of $\mathcal{F}$ with $f(y_0) = \alpha(0)$, $f(y_1) = \alpha(1)$, there exists a lifting $\beta \in \mathcal{F}^I$ of α with $\beta(i) = y_i$. More precisely, if we form the topos

$$\mathcal{E}^I \times_{(\mathcal{E} \times_S \mathcal{E})} (\mathcal{F} \times_S \mathcal{F})$$

of such triples (α, y_0, y_1) by pulling back $\mathcal{F} \times_S \mathcal{F} \xrightarrow{f \times f} \mathcal{E} \times_S \mathcal{E}$ along the map in (1), then the result can be stated as follows.

THEOREM 1. *Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a connected locally connected map of toposes. Then the canonical map*

$$\mathcal{F}^I \to \mathcal{E}^I \times_{(\mathcal{E} \times_S \mathcal{E})} (\mathcal{F} \times_S \mathcal{F})$$

given by the map $\mathcal{F}^I \to \mathcal{E}^I$ induced by f, and $\mathcal{F}^I \to \mathcal{F} \times_S \mathcal{F}$ as in (1), *is a stable surjection.*

For the special case of spatial toposes, or equivalently, for the case of spaces (in the generalized sense of e.g. **[JT]**, otherwise known as locales!) we can do slightly better.

Received by the editors December 3, 1985 and, in revised form, October 21, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 18B25, 18F10, 54D05, 54E50.

THEOREM 2. *Let $Y \xrightarrow{f} X$ be a connected locally connected map of (generalized) spaces. Then the canonical map*

$$Y^I \to X^I \times_{(X \times X)} (Y \times Y)$$

is an open surjection.

I conjecture that Theorem 1 can also be strengthened by replacing "stable surjection" with "open surjection."

Finally, it may be worthwhile to state the more down to earth case of metric spaces explicitly. Recall that an open map $Y \xrightarrow{f} X$ of *topological* spaces is locally 0-acyclic if Y has a basis of open sets which intersect all fibers of f in a connected (if nonempty) set (See [**MV**]). We will see that the following corollary is just a special case of Theorem 2, with the base topos S taken as the category of classical sets.

COROLLARY. *Let $Y \xrightarrow{f} X$ be a 0-acyclic map of complete separable metric spaces, and assume f has connected fibers. Let $Y^I \to X^I$ be the induced map of function spaces (with the compact-open topology), and let*

$$S = \{(\alpha, y_0, y_1) \mid \alpha\colon I \to X,\ \alpha(0) = f(y_0),\ \alpha(1) = f(y_1)\},$$

topologized as a subspace of $X^I \times Y \times Y$. Then the map

$$Y^I \to S, \qquad \beta \mapsto (f \circ \beta, \beta(0), \beta(1))$$

is an open surjection.

PRELIMINARIES, NOTATIONAL CONVENTIONS. This note is written as a sequel to [**MW**], and we assume that the reader is familiar with that paper. All the basic results and the notation that we use here can be found in §1 of [**MW**].

1. Reduction to the generic case. The only technically difficult thing to be proved is the following lemma.

1.1. LEMMA. *Let $Y \xrightarrow{f} I$ be a connected locally connected map of a space Y into the unit interval I in the base topos S, and let $y_0 \in f^{-1}(0)$, $y_1 \in f^{-1}(1)$ be two points of Y. Then there exists an open surjection $\mathcal{G} \to S$ such that in the topos $\mathcal{G}$, f has a section $I \xrightarrow{s} Y$ with $s(0) = y_0$, $s(1) = y_1$. In other words, there is a commutative diagram of toposes and geometric morphisms:*

$$\begin{array}{ccc} \mathcal{G} \times_S \{0,1\} & \xrightarrow{(y_0, y_1)} & Y \\ \big\downarrow & \overset{s}{\nearrow} & \big\downarrow f \\ \mathcal{G} \times_S I & \xrightarrow{\pi_2} & I \end{array}$$

This lemma will be proved in §2.

1.2. PROOF OF THEOREM 1. Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a connected, locally connected map of toposes. It suffices to show that for any map $\mathcal{G} \to \mathcal{E}^I \times_{(\mathcal{E} \times_S \mathcal{E})} (\mathcal{F} \times_S \mathcal{F})$ there exists an open surjection $\mathcal{H} \to \mathcal{G}$ and a map $\mathcal{H} \to \mathcal{F}^I$ such that

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{F}^I \\ \big\downarrow & & \big\downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{E}^I \times_{(\mathcal{E} \times_S \mathcal{E})} (\mathcal{F} \times_S \mathcal{F}) \end{array} \tag{1}$$

commutes. By working in $\mathcal{G}$, i.e. by replacing $\mathcal{S}$ by $\mathcal{G}$, we may assume $\mathcal{G} = \mathcal{S}$ (the map $\mathcal{F} \to \mathcal{E}$ remains connected locally connected after this change of base). So suppose we are given a map $I \xrightarrow{\alpha} \mathcal{E}$ and two points x_0, x_1 of $\mathcal{F}$, with $f(x_i) = \alpha(i)$. Let $Y \xrightarrow{g} \mathcal{F}$ be a connected locally connected map, where Y is a space (identified with the corresponding topos $Sh(Y)$), as in [**MW**, §1.6]. By extending $\mathcal{S}$, i.e. by replacing $\mathcal{S}$ with $\mathcal{S}'$ where $\mathcal{S}' \to \mathcal{S}$ is an open surjection, we may assume that in $\mathcal{S}$ there are points $y_0, y_1 \in Y$ with $g(y_i) = x_i$. Consider the pullback of toposes over $\mathcal{S}$:

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & Y \\ & & \downarrow g \\ \Big\downarrow h & & \mathcal{F} \\ & & \downarrow f \\ I & \xrightarrow{\alpha} & \mathcal{E} \end{array}$$

(Z is spatial over $\mathcal{S}$, since Y is spatial over $\mathcal{S}$, so a fortiori over $\mathcal{E}$.) Let z_0, z_1 be the points of Z with $\beta(z_i) = y_i$, $h(z_i) = i$. By Lemma 1.1 there exists an open surjection $\mathcal{H} \to \mathcal{S}$ such that in $\mathcal{H}$ there is a map $I \xrightarrow{s} Z$ with $h \circ s = \mathrm{id}$ and $s(i) = z_i$. Composing with $g \circ \beta$ and transposing, this gives the map $\mathcal{H} \to \mathcal{F}^I$ required in (1). □

1.3. Proof of Theorem 2. Let $Y \xrightarrow{f} X$ be a connected locally connected map of spaces. It follows from Theorem 1 that

$$Y^I \to X^I \times_{(X \times X)} (Y \times Y) = S$$

is a stable surjection. To prove that $Y^I \to S$ is in fact open, one shows that the image of a basic open of the form $\bigwedge_{i=0}^{n-1} [(p_i, p_{i+1}), U_i]$, where $p_0 = 0 < p_1 < \cdots < p_n = 1$ are rationals and $(U_0, \ldots, U_{n-1})$ is a chain of opens in Y, is the open $(\bigwedge_{i=0}^{n-1} [(p_i, p_{i+1}), f(U_i)]) \times_{(X \times X)} (U_0 \times U_{n-1})$ of S. This is completely similar to [**MW**, §2.7]. □

1.4. Proof of the Corollary. It is observed in [J] that an open map $X \to Y$ of topological spaces is locally 0-acyclic iff it is locally connected as a map of (generalized) spaces, or equivalently, as a map of toposes. If X and Y are complete separable metric spaces in Sets, then the corresponding locales $\mathcal{O}(X)$, $\mathcal{O}(Y)$ are countably presented, and hence so are the locales corresponding to the exponentials X^I, Y^I, and the pullback $X^I \times_{(X \times X)} (Y \times Y)$ as generalized spaces. Consequently, these generalized spaces have enough points [**MR**], i.e. they coincide with the corresponding topological spaces. By these general facts, the corollary follows immediately from Theorem 2. □

1.5. Remark on path-connectedness. Note that by taking $X = 1$ in Theorem 2, it follows that if Y is a connected locally connected space, the map $Y^I \to Y \times Y$ is an open surjection. This is the special case proved in [**MW**]. As explained there, the corresponding fact for toposes, stated in the first lines of this paper, follows easily (cf. [**MW**, §2.1]).

Similarly, by taking $X = 1$ in the Corollary, we obtain the old result of K. Menger and R. L. Moore, saying that all connected and locally connected complete separable metric spaces are path-connected (for references, see [**MW**]). □

2. The generic case. We will now prove Lemma 1.1. A crucial preliminary result is the following.

2.1. PROPOSITION. *Let $Y \xrightarrow{f} X$ be a locally connected map of spaces, and let $\mathbf{P}$ be a presentation (a poset with a stable covering system) for X. Then there exists a presentation $\mathbf{Q}$ for Y such that taking direct images induces a function*

$$\mathbf{Q} \xrightarrow{f(-)} \mathbf{P}, \qquad U \mapsto f(U) = \exists_f(U)$$

with the following property: Given a cover $\{U_\alpha\}_\alpha$ of $U \in \mathbf{Q}$, and $V \in \mathbf{P}$ with $V \leq f(U_\alpha) \wedge f(U_\beta)$ for two given indices α and β, there exists a cover $\mathcal{W}$ of V in $\mathbf{P}$ such that for each $W \in \mathcal{W}$ there is a chain $U_\alpha = U_{\alpha_1}, \ldots, U_{\alpha_n} = U_\beta$ from U_α to U_β in $\mathbf{Q}$ with $W \leq f(U_{\alpha_i} \wedge U_{\alpha_{i+1}})$ for $i = 1, \ldots, n-1$ (or more precisely, since $\mathbf{P}$ may not have meets, $W \leq f(\tilde{U}_i)$ for some $\tilde{U}_i \in \mathbf{P}$, $\tilde{U}_i \leq U_{\alpha_i} \wedge U_{\alpha_{i+1}}$). Moreover, if $Y \xrightarrow{f} X$ is also a connected map of spaces, we may take $f^{-1}(V) \in \mathbf{Q}$ for each $V \in \mathbf{P}$, so $Y \xrightarrow{f} X$ is determined by the adjoint pair $\mathbf{Q} \underset{f^{-1}}{\overset{f(-)}{\rightleftarrows}} \mathbf{P}$.

PROOF. This is really a special case of Lemma 2.5 from [**M**]. But it can also be proved directly, by starting with a molecular presentation of Y, where Y is considered as an internal space in $Sh(\mathbf{P})$. □

2.2. REMARK. Intuitively, the elements of $\mathbf{Q}$ are the opens of Y all of whose nonempty fibers are connected. Clearly, we cannot assume that $\mathbf{Q}$ is closed under unions of chains, i.e. we need not have $(U, U' \in \mathbf{Q})$ and $\mathrm{Pos}(U \wedge U') \Rightarrow (U \vee U' \in \mathbf{Q})$. It is easy to see, however, that when $\mathbf{Q}$ comes from a molecular presentation of Y in $Sh(\mathbf{P})$, it has the following property.

(1) If $U, V \in \mathbf{Q}$ with $\mathrm{Pos}(U \wedge V)$ and $f(U \wedge V) = f(U) \wedge f(V)$, then $U \vee V \in \mathbf{Q}$.
Moreover, it will hold that

(2) if $U \in \mathbf{Q}$, $V \in \mathbf{P}$ and $V \leq f(U)$, then $U \wedge f^{-1}(V) \in \mathbf{Q}$. □

2.3. Proof of Lemma 1.1. Let $Y \xrightarrow{f} I$ be a connected, locally connected map of spaces in $\mathcal{S}$, and let $\mathbf{P}$ be the presentation of I by rational intervals. (As in [**MW**], we take (p, q) to stand for $[p, q)$ if $p = 0$, and for $(p, q]$ if $q = 1$.) Let $\mathbf{Q}$ be a presentation for Y as in 2.1, coming from an internal molecular presentation of Y in $Sh(\mathbf{P})$; so (1) and (2) of 2.2 will hold. Thus $f: Y \to I$ is induced by the adjoint pair

$$\mathbf{Q} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \mathbf{P}$$

and $\mathbf{Q}$ is molecular since $\mathbf{P}$ is (cf. also [**M**, lemma in §4]).

We now proceed in the style of [**MW**, §2.4 and 2.5]. First of all, by replacing $\mathcal{S}$ by an (open surjective) extension $\mathcal{S}' \to \mathcal{S}$ of $\mathcal{S}$, we may assume that we have the following enumerations:

(i) for each $U \in \mathbf{Q}$, $(\mathcal{U}_n(U) \mid n \in \mathbf{N})$ enumerates the covers of U in $\mathbf{Q}$;

(ii) $(N_n(y_0) \mid n \in \mathbf{N})$ and $(N_n(y_1) \mid n \in \mathbf{N})$ enumerate the elements of $\mathbf{Q}$ which contain y_0 and y_1, respectively,
just as in [**MW**, §2.4].

We now build a tree T of pairs of finite sequences

$$((V_1^m, \ldots, V_{k(m)}^m)_{m \leq n}, (p_0^m, \ldots, p_{k(m)}^m)_{m \leq n}),$$

where each $(V_1^m, \ldots, V_{k(m)}^m)$ is a chain in $\mathbf{Q}$ from y_0 to y_1, and $0 = p_0^m < p_1^m < \cdots < p_{k(m)}^m = 1$ are rationals, such that

(a) for each $m < n$, $\{p_0^m, \ldots, p_{k(m)}^m\} \subset \{p_0^{m+1}, \ldots, p_{k(m+1)}^{m+1}\}$, and the chain $(V_1^{m+1}, \ldots, V_{k(m+1)}^{m+1})$ refines $(V_1^m, \ldots, V_{k(m)}^m)$ accordingly, i.e. for $1 \leq i \leq k(m)$, $1 < j < k(m+1)$, $(p_{j-1}^{m+1}, p_j^{m+1}) \subset (p_{i-1}^m, p_i^m) \Rightarrow V_j^{m+1} \leq V_i^m$;

(b) for each $m' < m \leq n$ and each $j \leq k(m)$, V_j^m is contained in an element of $\mathcal{U}_l(V_i^{m'})$ for each $l \leq n$, where $i \leq k(m')$ is the index with $(p_{j-1}^m, p_j^m) \subset (p_{i-1}^{m'}, p_i^{m'})$;

(c) V_1^m is contained in $N_l(z_0)$ for each $l \leq m$, and $V_{k(m)}^m$ is contained in $N_l(z_1)$ for each $l \leq m$;

(d) given $1 < i \leq k(m)$, suppose $p_{j-1}^{m+1} = p_{i-1}^m$, $p_{j+k}^{m+1} = p_i^m$ (so $(p_{j-1}^{m+1}, p_j^{m+1}), \ldots, (p_{j+k-1}^{m+1}, p_{j+k}^{m+1})$ are all contained in (p_{i-1}^m, p_i^m), and accordingly $V_j^{m+1} \vee \cdots \vee V_{j+k}^{m+1} \leq V_i^m$ (see (a) above)). Then $V_j^{m+1} \leq V_{i-1}^m$ (unless $j + k = k(m+1)$, i.e. $i = k(m)$);

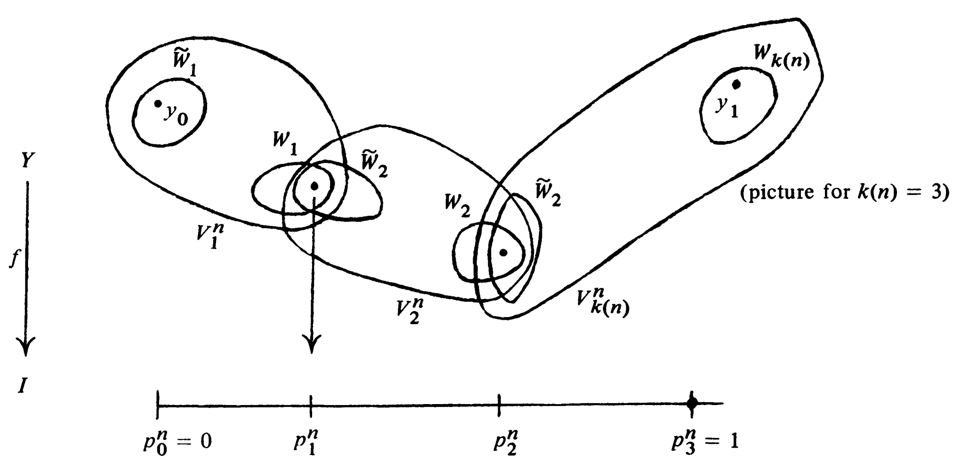
(e) for $1 \leq i \leq j \leq k \leq k(m)$, $f(V_i^m \vee \cdots \vee V_j^m) \wedge f(V_{j+1}^m \vee \cdots \vee V_k^m) = f((V_i^m \vee \cdots \vee V_j^m) \wedge (V_{j+1}^m \vee \cdots \vee V_k^m))$;

(f) for $i = 1, \ldots, k(m)$, $[p_{i-1}^m, p_i^m] \leq f(V_i^m)$, i.e., there are rationals r, r', $r < p_{i-1}^m < p_i^m < r'$, with $(r, r') \leq f(V_i^m)$.

We claim that by molecularity of $\mathbf{P}$ and $\mathbf{Q}$, and by 2.2, 2.3 above, any pair of sequences satisfying (a)–(f) can be extended to a longer one. To see this, suppose we are given $((V_1^m, \ldots, V_{k(m)}^m)_{m \leq n}, (p_0^m, \ldots, p_{k(m)}^m)_{m \leq n})$ as above. Cover each V_j^n $(j = 1, \ldots, k(n))$ by a common refinement $\mathcal{W}_j$ of the covers $\mathcal{U}_{n+1}(V_{i_m}^m)$ $(m \leq n, i_m \leq k(m)$ the index such that $(p_{j-1}^n, p_j^n) \subset (p_{i_m-1}^m, p_{i_m}^m))$. Choose $\tilde{W} \ni y_0$ in $\mathbf{Q}$ such that $\tilde{W}_1 \leq$ some element of $\mathcal{W}_1$ and $\tilde{W}_1 \leq N_{n+1}(y_0)$, and similarly choose $W_{k(n)} \ni y_1$ in $\mathbf{Q}$ such that $W_{k(n)} \leq$ some element of $\mathcal{W}_{k(n)}$ and $W_{k(n)} \leq N_{n+1}(y_0)$. $\tilde{W}_1$ and $W_{k(n)}$ having been chosen, also fix for each of $j < k(n)$ a $W_j \in \mathcal{W}_j$ and a $\tilde{W}_{j+1} \in \mathcal{W}_{j+1}$ which have positive intersection. This can be done by molecularity of $\mathbf{Q}$. Moreover, we choose W_j and $\tilde{W}_{j+1}$ in such a way that there is a $U_j \in \mathbf{Q}$ with

$$(1) \qquad U_j \leq W_j \quad \text{and} \quad U_j \leq \tilde{W}_{j+1} \quad \text{and} \quad p_j^n \in f(U_j),$$

since by (e) and (f), $p_j^n \in f(V_j^n \wedge V_{j+1}^n)$.



So $0 = p_0^n \in f(\tilde{W}_1)$, $p_j^n \in f(W_j \wedge \tilde{W}_{j+1})$ $(0 < j < k(n))$, and $1 = p_{k(n)}^n \in f(W_{k(n)})$. Since f is an open surjection, there are for each j, $1 \leq j \leq k(n)$, opens

$$\tilde{W}_j = O_1^j, \ldots, O_{u_j}^j = W_j$$

in $\mathcal{W}_j$ such that $(f(O_1^j), \ldots, f(O_{u_j}^j))$ is a chain in $\mathbf{P}$ from p_{j-1}^n to p_j^n.

By Proposition 2.1 we can find rational intervals $(q_1^j, r_1^j), \ldots, (q_{u_j-1}^j, r_{u_j-1}^j)$ with

$$p_{j-1}^n < q_1^j < r_1^j < \cdots < q_{u_j-1}^j < r_{u_j-1}^j < p_j^n \qquad (j = 1, \ldots, k(n)),$$

and for each $s < u_j$ a chain

$$O_s^j = O_{s,1}^j, \ldots, O_{s,t_{s,j}}^j = O_{s+1}^j \tag{2}$$

in $\mathcal{W}_j$ from O_s^j to O_{s+1}^j $(s = 1, \ldots, u_j - 1)$ with $(q_s^j, r_s^j) \leq f(O_{s,t}^j \wedge O_{s,t+1}^j)$ for each $s < u_j$ and $t < t_{s,j}$.

By 2.2(2), we can refine this chain (2) by letting

$$\begin{aligned}
A_{s,t}^j &= O_{s,t}^j \wedge f^{-1}(q_{s,t}^j, r_{s,t}^j), \qquad 1 < t < t_{s,j},\ 1 \leq s < u_j, \\
A_{1,1}^j &= O_1^j \wedge f^{-1}(p_{j-1}^n, r_1^j) = \tilde{W}_j \wedge f^{-1}(p_{j-1}^n, r_1^j), \\
A_{s,t_s}^j &= O_s^j \wedge f^{-1}(q_s^j, r_{s+1}^j) = A_{s+1,1}^j, \qquad 1 < s < u_j - 1, \\
A_{u_j-1,t_{u_j-1}}^j &= O_{u_j}^j \wedge f^{-1}(q_{u_j-1}^j, p_j^n) = W_j \wedge f^{-1}(q_{u_j-1}^j, p_j^n).
\end{aligned}$$

So now we have for each $j = 1, \ldots, k(n)$ a chain

$$A_{1,1}^j, A_{1,2}^j, \ldots, A_{1,t_1}^j = A_{2,1}^j, A_{2,2}^j, \ldots, A_{u_j-1,t_{u_j-1}}^j \tag*{$(3)^j$}$$

with $A_{1,1}^j \leq \tilde{W}_j, A_{u_j-1,t_{u_j-1}}^j \leq W_j$.

To define $(V_1^{n+1}, \ldots, V_{k(n+1)}^{n+1})$ we take the chains $(3)^j$, one after the other, but with an open inserted, to "glue" the $(3)^j$-chain to the $(3)^{j+1}$ one, as follows. Fix j, $1 \leq j < k(n)$, and choose rationals a_j, b_j with

$$r_{u_j-1}^j < a_j < p_j^n < b_j < q_1^{j+1} \quad \text{and} \quad (a_j, b_j) \leq f(U_j), \tag{4}$$

$$B_j = U_j \wedge f^{-1}(a_j, b_j), \qquad 1 \leq j < k(n),$$

and define $(V_1^{n+1}, \ldots, V_{k(n+1)}^{n+1})$ to be the chain

$$(3)^1, B_1, B_1, (3)^2, B_2, B_2, \ldots, (3)^{k(n)-1}, B_{k(n)-1}, B_{k(n)-1}, (3)^{k(n)}, \tag{5}$$

where $(3)^j$ abbreviates the chain in formula $(3)^j$ above.

Finally, we define the rationals $(p_0^{n+1}, \ldots, p_{k(n+1)}^{n+1})$, refining the sequence $(p_0^n, \ldots, p_{k(n)}^n)$. For each $j = 1, \ldots, k(n)$ and $s = 1, \ldots, u_j - 1$, we have a chain $(A_{s,t}^j | 1 < t < t_{s,j})$ over the interval (q_s^j, r_s^j), and we subdivide this interval into $t_{s,j} - 2$ pieces accordingly: take rationals $c_{s,t}^j$: $1 < t < t_{s,j} - 1$ with

$$q_s^j < c_{s,2}^j < \cdots < c_{s,t_{s,j}-2}^j < r_s^j. \tag*{$(6)_s^j$}$$

To refine $(p_0^n, \ldots, p_{k(n)}^n)$, we replace (p_{j-1}^n, p_j^n) for $1 < j < k(n)$ by

$$(p_{j-1}^n, b_{j-1}, (6)_1^j, \ldots, (6)_{u_j-1}^j, a_j, p_j^n), \tag*{$(7)_j$}$$

where $(6)^j_s$ abbreviates the $(t_{s,j}-1)$-tuple in formula $(6)^j_s$ above; for $j=1, j=k(n)$, we replace (p^n_0, p^n_1) and $(p^n_{k(n)-1}, p^n_{k(n)})$ respectively by a sequence as in $(7)_j$, but without the a_j and the b_{j-1}.

This, finally, defines $((V^{n+1}_1, \dots, V^{n+1}_{k(n+1)}), (p^{n+1}_0, \dots, p^{n+1}_{k(n+1)}))$. We leave it to the reader to check that (a)–(f) are indeed satisfied.

After this rather tedious part of the proof, we can finish quite straightforwardly, just as in [**MW**]. By Lemma C of §1.6 of that paper, there is an open surjection $\mathcal{G} \to \mathcal{S}$ in which the tree T has an infinite branch. We replace $\mathcal{S}$ by $\mathcal{G}$, and work in $\mathcal{G}$ with this fixed branch:

$$((V^m_1, \dots, V^m_{k(m)})_{m\in\mathbf{N}}, (p^m_0, \dots, p^m_{k(m)})_{m\in\mathbf{N}}).$$

Using this branch, we can define a map $s: I \to Y$ of generalized spaces by the function

$$s^*: \mathbf{Q} \to \mathcal{O}(I), \quad S^*(U) = \bigvee\{(p^m_j, p^m_{j'}) \mid j < j' \text{ and } V^m_{j+1} \vee \dots \vee V^m_{j'} \le U\}.$$

Then just as in [**MW**] it follows easily from (a)–(e) that s^* preserves $\wedge$ and $\bigvee$, and that $s(0) = y_0$, $s(1) = y_1$. (By condition (e), sups of the form $V^m_{j+1} \vee \dots \vee V^m_j$ are in $\mathbf{Q}$ (cf. 2.2)–this is used to prove that s^* preserves $\wedge$. In [**MR**], sups of chains were automatically elements of the presentation.)

Finally, we have to show that s is a section of f, i.e., $f \circ s$ is the identity on I. But by condition (f), we have $\forall U \in \mathbf{Q}\ s^{-1}(U) = s^*(U) \le f(U)$, so in particular

$$\forall V \in \mathbf{P}\ (f \circ s)^{-1}(V) = s^{-1} f^{-1}(V) \le f f^{-1}(V) = V. \tag{8}$$

But in any topos, the (formal) unit interval I is a T_1-space, in the appropriate sense of generalized spaces (see e.g. [**F**]), i.e.

$$\text{for any pair } X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} I \text{ of maps of generalized spaces,} \tag{9}$$

$$\forall V \in \mathcal{O}(I)\ \varphi^{-1}(V) \le \psi^{-1}(V) \text{ implies } \varphi = \psi.$$

So for our particular case we conclude that $f \circ s = \text{id}$.

This completes the proof of Lemma 1.1.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, QUEBEC, CANADA, H3A 2K6

Current address: Department of Mathematics, University of Chicago, Chicago, Illinois 60637