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ONE-RELATOR QUOTIENTS AND FREE PRODUCTS OF CYCLICS

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ABSTRACT. It is proven that the Freiheitssatz holds for all one-relator products of cyclic groups if the relator is cyclically reduced and a proper power. The method of proof involves representing such groups in $\mathrm{PSL}_2(\mathbf{C})$ and is a refinement of a technique of Baumslag, Morgan and Shalen. The technique allows the extension of the Freiheitssatz result to many additional one-relator products.

1. Introduction. A one-relator product of a family of groups (A_i), $i \in I$, is the quotient $(*A_i)/N(R)$ where R is a cyclically reduced word and $N(R)$ is its normal closure in the free product $*_i A_i$. R is called the relator. A one-relator group is just a one-relator product of free groups. In general one-relator products share many properties with one-relator groups [**3**].

In the present paper we give a version of the Freiheitssatz for a certain class of one-relator products of cyclic groups. In particular we show that if each A_i is cyclic and the relator is a proper power then the subgroup generated by any proper subset of the generators is the obvious free product of cyclics. This answers a conjecture of J. Howie [**3**]. The method of proof involves representing such groups in $\mathrm{PSL}_2(\mathbf{C})$ and is a refinement of a technique of Baumslag, Morgan and Shalen [**1**]. This technique was used in [**1**] to investigate the class of generalized triangle groups which in the present context can be described as one-relator products of two cyclic groups. The technique actually yields a stronger result—a Freiheitssatz for one-relator products of groups which admit faithful representation in $\mathrm{PSL}_2(\mathbf{C})$.

2. Freiheitssatz. Our main result is the following.

THEOREM 1. *Suppose the group G is given by*

$$G = \langle a_1, a_2, \ldots, a_n; a_1^{e_1} = a_2^{e_2} = \cdots = a_n^{e_n} = R^m(a_1, a_2, \ldots, a_n) = 1\rangle$$

with $n \geq 2$, $m \geq 2$, $e_i = 0$ or $e_i \geq 2$ for $i = 1, \ldots, n$ and $R(a_1, a_2, \ldots, a_n)$ a cyclically reduced word in the free product on $a_1, a_2, \ldots, a_n$ which involves all $a_1, \ldots, a_n$. Then

$$\langle a_1, a_2, \ldots, a_{n-1}\rangle = \langle a_1; a_1^{e_1}\rangle * \langle a_2; a_2^{e_2}\rangle * \cdots * \langle a_n; a_n^{e_n}\rangle,$$

i.e. $\langle a_1, \ldots, a_{n-1}\rangle$ is the free product of cyclics of the obvious orders.

PROOF. The proof follows the technique of [**1**]. Without loss of generality we can assume $e_i \geq 2$ for $i = 1, \ldots, n$.

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First we choose projective matrices $A_1, A_2, \ldots, A_{n-1}$ in $\mathrm{PSL}_2(\mathbf{C})$ such that the subgroup generated by these is the appropriate free product of cyclics. That is

$$\langle A_1, A_2, \ldots, A_{n-1}\rangle = \langle A_1; A_1^{e_1}\rangle * \langle A_2; A_2^{e_2}\rangle * \cdots * \langle A_{n-1}; A_{n-1}^{e_{n-1}}\rangle$$

and so $\langle A_1, A_2, \ldots, A_{n-1}\rangle$ faithfully represents the free product of cyclics $\langle a_1, \ldots, a_{n-1}; a_1^{e_1} = \cdots = a_{n-1}^{e_{n-1}} = 1\rangle$.

We will attempt to determine a projective matrix $A_n \in \mathrm{PSL}_2(\mathbf{C})$ of appropriate order e_n so that the subgroup generated by $A_1, A_2, \ldots, A_n$ provides a representation of G in $\mathrm{PSL}_2(\mathbf{C})$. Since the image of $a_1, a_2, \ldots, a_{n-1}$ will then be a free product of cyclics (each of maximal possible order), $\langle a_1, a_2, \ldots, a_{n-1}\rangle$ will also be the appropriate free product of cyclics.

Choose

$$A_n = \begin{pmatrix} w & wt - w^2 - 1 \\ 1 & t - w \end{pmatrix}$$

where $t = 2\cos(\pi/e_n)$ and w is to be determined. Since $\mathrm{tr}(A_n) = 2\cos(\pi/e_n)$ we have $A_n^{e_n} = 1$ in $\mathrm{PSL}_2(\mathbf{C})$.

Consider the relator word $R(a_1, a_2, \ldots, a_n)$. Substituting $A_1, A_2, \ldots, A_n$ in R we obtain the matrix

$$R(A_1, A_2, \ldots, A_n) = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

where f_1, f_2, f_3, f_4 are polynomials in the coefficients of $A_1, A_2, \ldots, A_n$. Considering w as the only unknown, f_1, f_2, f_3, f_4 are then polynomials in w.

If $\mathrm{tr}(R(A_1, A_2, \ldots, A_n)) = f_1 + f_4$ is not a constant polynomial in w then the polynomial equation

$$f_1(w) + f_4(w) = 2\cos(\pi/m)$$

can be solved for w.

For this choice of w in A_n we would have $R^m(A_1, A_2, \ldots, A_m) = 1$ because of the trace. Therefore the subgroup generated by $A_1, A_2, \ldots, A_n$ provides a representation of G in $\mathrm{PSL}_2(\mathbf{C})$ with $A_1, A_2, \ldots, A_{n-1}$ generating the appropriate free product of cyclics.

What is left in order to complete the proof is to show that there is a choice of $A_1, A_2, \ldots, A_{n-1}$ such that $\mathrm{tr}(R(A_1, A_2, \ldots, A_n))$ is nonconstant in w.

Since $R(a_1, a_2, \ldots, a_n)$ is cyclically reduced we can write the relator R without loss of generality as

$$R(A_1, A_2, \ldots, A_n) = B_1 A_n^{t_1} B_2 \cdots B_k A_n^{t_k}$$

with B_i nontrivial words in $A_1, A_2, \ldots, A_{n-1}$ (hence nonidentity matrices) and $1 \le t < e_n$.

Suppose $B_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \ne \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Then $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) B_1 \left(\begin{smallmatrix} 1 & 0 \\ -y & 1 \end{smallmatrix}\right)$ has lower left entry $c + ya - yd - y^2 b$ which is nonzero for all but finitely many choices of y. Similarly for $B_2, \ldots, B_k$. Then by conjugating all the $B_1, B_2, \ldots, B_k$ by a suitable $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$ if necessary we can assume that the lower left entries of $B_1, B_2, \ldots, B_k$ are nonzero. By considering the diagonalization of A_n we see that $A_n^{t_i}$ has the form

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

where g_1, g_2, g_3, g_4 are polynomials in w of respective degrees $1, 2, 0$ and 1. It follows that each factor $B_i A_n^{t_i}$ in the above expression for the relator R must have the form

$$\begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$$

where h_1, h_2, h_3, h_4 are also polynomials in w. Since each B_i has nonzero lower left entry we must have the degree of h_4 is exactly 2 while the degrees of h_1, h_2, h_3 are $\leq 1, \leq 2, \leq 1$ respectively. Using an induction on $k \geq 1$ we can conclude that in R, f_1 and f_3 are of degree $< 2k$, f_2 is of degree $\leq 2k$ and f_4 is of degree exactly $2k$. This guarantees that $\operatorname{tr}(R(A_1, \ldots, A_n)) = f_1 + f_4$ is nonconstant in w.

For the proper power case as in the above theorem the Freiheitssatz was known to hold if the order of the relator is 4 or greater—that is R is cyclically reduced and $m \geq 4$. It was conjectured by Howie [**3**] that this could be weakened to $m \geq 2$. Thus Theorem 1 answers Howie's conjecture for the class of one-relator products of cyclics.

The proof of Theorem 1 depended on the fact that free products of cyclics can be faithfully represented in $\mathrm{PSL}_2(\mathbf{C})$. A straightforward extension then yields the following stronger result. Our main Theorem 1 can actually be considered as a special case of Theorem 2.

THEOREM 2. *Let $G = (A * B)/N(R^m)$ where A and B are groups admitting faithful representations in $\mathrm{PSL}_2(\mathbf{C})$, and R is a cyclically reduced word in the free product of A and B of length ≥ 2 and $m \geq 2$. Then G admits a representation $\rho\colon G \to \mathrm{PSL}_2(\mathbf{C})$ such that $B \to G \xrightarrow{\rho} \mathrm{PSL}_2(\mathbf{C})$ and $A \to G \xrightarrow{\rho} \mathrm{PSL}_2(\mathbf{C})$ are faithful and $\rho(R)$ has order m. In particular $A \to G$ and $B \to G$ are injective—that is the Freiheitssatz holds.*

PROOF. Write the relator as

$$R = a_1 b_1 \cdots a_k b_k \quad \text{with } a_i \in A, b_i \in B.$$

Since R is cyclically reduced of length ≥ 2 we can assume that $a_i \neq 1$ and $b_i \neq 1$.

Choose faithful representations $\rho_A\colon A \to \mathrm{PSL}_2(\mathbf{C})$ and $\rho_B\colon B \to \mathrm{PSL}_2(\mathbf{C})$ such that (after suitable conjugation if necessary)

$$\rho_A(a_i) = \begin{pmatrix} * & x_i \\ * & * \end{pmatrix} \quad \text{with } x_i \neq 0,$$

$$\rho_B(b_i) = \begin{pmatrix} * & y_i \\ * & * \end{pmatrix} \quad \text{with } y_i \neq 0,$$

for all $i = 1, \ldots, k$.

For $w \in \mathbf{C}$ let ρ_A^w denote the representation ρ_A conjugated by $\left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$. That is,

$$\rho_A^w(a) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \rho_A(a) \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}.$$

Considering w as a variable define

$$T(w) = \operatorname{tr}(\rho_A^w(a_1)\rho_B(b_1) \cdots \rho_A^w(a_k)\rho_B(b_k)).$$

Then $T(w)$ is a polynomial in w. As in the proof of Theorem 1 the coefficient of w^{2k} is nonzero $(= \pm x_1 y_1 \cdots x_k y_k)$ because of the choices of ρ_A and ρ_B.

Then again as in the proof of Theorem 1 there exists a w_0 with

$$T(w_0) = 2\cos(\pi/m).$$

Now define the representation $\rho\colon G \to \mathrm{PSL}_2(\mathbf{C})$ by $\rho|_A = \rho_A^{w_0}$ and $\rho|_B = \rho_B$. Then $\rho(R)$ has trace $T(w_0) = 2\cos(\pi/m)$ and so $\rho(R)$ has the correct order m. Further $\rho|_A = \rho_A^{w_0}$ is faithful on A and similarly $\rho|_B$ is faithful on B. Thus ρ is the desired representation.

Theorem 2 allows us to greatly extend the class of one-relator products for which the Freiheitssatz holds. If G is a group of the form $(A * B)/N(R^m)$ with R and m as in the theorem then $A \to G$ and $B \to G$ are injective (the Freiheitssatz holds) whenever A and B are Kleinian groups, Fuchsian groups, Surface groups, free abelian groups of rank 2, free metabelian groups of rank 2 or fixed combinations of any of these. This is a consequence of the fact that each of these types of groups can be faithfully represented in $\mathrm{PSL}_2(\mathbf{C})$.

Using essentially the same technique as the proof of Theorem 1, Baumslag, Morgan and Shalen investigated the generalized triangle groups. These are groups of the form $\langle a, b; a^m = b^p = S^n(a,b) = 1\rangle$ with $S(a,b)$ a word in a and b. In the context of Theorem 1 these can be described as one-relator products of two cyclic groups. In particular they obtained the following two results.

THEOREM A (BAUMSLAG, MORGAN AND SHALEN [1]). *Let $G = \langle a, b; a^m = b^p = S^n(a,b) = 1\rangle$ with $m, n, p \geq 2$. Then $G \neq \{1\}$.*

THEOREM B (BAUMSLAG, MORGAN AND SHALEN [1]). *Let $G = \langle a, b; a^m = b^p = S^n(a,b) = 1\rangle$ and let $W(G) = 1/m + 1/p + 1/n$. Then*

(1) *If $W(G) \leq 1$ then G is infinite,*

(2) *If $W(G) < 1$ then G contains a free subgroup of rank 2.*

Theorem B mirrors the situation for the ordinary triangle groups—that is where $S(a,b) = ab$.

Part (2) of Theorem B raises the question of the nature of the nonabelian free subgroups of the generalized triangle groups. A result of Ree and Mendelsohn [6] shows that if a and b both have infinite order ($m = p = 0$) above and S is a proper power involving both a and b then for sufficiently large t the elements a and b^t generate a free subgroup rank 2. Our methods allow us to extend this. First if one generator has infinite order:

COROLLARY 1. *Let $G = \langle a, b; a^m = S^n(a,b) = 1\rangle$ with $m \geq 2$, $n \geq 3$ and $S(a,b)$ a cyclically reduced word in the free product on a and b which involves both a and b. Then there exists a generating pair $\{u, v\}$ of G and a sufficiently large integer t such that $\langle u^t, v^t\rangle$ is a free group of rank 2.*

From Theorem 1 we can find a representation of G in $\mathrm{PSL}_2(\mathbf{C})$ where the image of a has order $m \geq 2$, the image of S has order n and the image of b has infinite order. The proof of Corollary 1 then involves a long technical argument on the possible types of images of a and b and their fixed points as mappings of the complex plane. This type of situation was handled in [7]. The details of the proof of Corollary 1 will appear elsewhere.

Using the same type of analysis we can handle the case where both generators have finite order and the relator is the commutator word $S(a,b) = [a,b]^n$.

COROLLARY 2. *Let* $G = \langle a, b; a^m = b^p = [a,b]^n = 1\rangle$ *with* $m = 0$ *or* $m \geq 2$, $p = 0$ *or* $p \geq 2$, $m \leq p$ *and* $n = 0$ *or* $n \geq 2$. *Suppose one of the following holds.*

(i) $n = 0$ *and* $p \geq 3$,
(ii) $m = 0$,
(iii) $p = 0$,
(iv) $m = 3$ *and* $n \geq 3$ *or* $m = 3$ *and* $p \geq 4$,
(v) $m = 2, p \geq 3$ *and* $n \geq 4$ *or* $m = 2$, $p \geq 4$ *and* $n \geq 3$,
(vi) $m = 2$ *and* $p \geq 5$.

Then there exists a generating pair $\{u, v\}$ *of* G *and a sufficiently large integer* t *such that* $\langle u^t, v^t\rangle$ *is a free group of rank* 2.

The proof of Corollary 2 depends, as in the last corollary, on the nature of the possible images of the generators of G in $\mathrm{PSL}_2(\mathbf{C})$. The details will appear with those of Corollary 1.

Finally as a consequence of Theorem 1 we can give a relatively simple proof of the following theorem of Fischer, Karrass and Solitar [**2**].

COROLLARY 3 (FISCHER, KARRASS, SOLITAR). *Let* $G = \langle a_1, a_2, \ldots, a_n; R^m(a_1, \ldots, a_n)\rangle$ *with* $m \geq 2$*—that is* G *is a one-relator group with torsion. Then* G *contains a normal torsion-free subgroup of finite index.*

PROOF. Without loss of generality we can assume that R is cyclically reduced and involves all $a_1, \ldots, a_n$. Let $\rho\colon G \to \mathrm{PSL}_2(\mathbf{C})$ be the representation constructed as in Theorem 1 so that $\rho|_{\langle a_1,\ldots,a_{n-1}\rangle}$ and $\rho|_{\langle a_n\rangle}$ are faithful and $\rho(R(a_1, \ldots, a_n))$ has order m.

Since G is finitely generated, $\rho(G)$ is a finitely generated subgroup of $\mathrm{PSL}_2(\mathbf{C})$. From a theorem of Selberg [**8**], $\rho(G)$ contains a normal torsion-free subgroup of finite index. Call this subgroup H. H then lifts to a normal subgroup $\overline{H}$ of finite index in G. We claim $\overline{H}$ is also torsion-free. If $g \in \overline{H}$ and g has finite order then $g \in \operatorname{Ker}\rho$ since H is torsion-free. But the elements of G of finite order are all conjugates of powers of the relator R [**5**]. Since $\rho(R)$ has order m it follows that $\operatorname{Ker}\rho$ is torsion-free. Therefore $g = 1$ and $\overline{H}$ is torsion-free.

Using a recent result of J. Howie [**4**] essentially the same proof gives the above result for one-relator quotients of locally indicable groups which admit faithful representations in $\mathrm{PSL}_2(\mathbf{C})$. Recall that a group G is locally indicable if every finitely generated nontrivial subgroup of G has an infinite cyclic homomorphic image. In particular,

COROLLARY 4. *Let* $G = (A * B)/N(R^m)$ *where* A *and* B *are finitely generated locally indicable groups which admit faithful representations in* $\mathrm{PSL}_2(\mathbf{C})$. *Suppose* R *is cyclically reduced in the free product* $A * B$ *of length* ≥ 2 *and not a proper power in* $A * B$. *Then if* $m \geq 2$, G *contains a torsion-free normal subgroup of finite index.*

The proof depends as in Corollary 3 on the fact that elements of finite order are conjugates of powers of R. This was shown for groups satisfying the hypotheses of the corollary in [**4**]. We note also that Kleinian groups of the second kind are locally indicable and therefore the corollary above holds for one-relator products of finitely generated Kleinian groups of the second kind. We thank the referee for pointing this out to us.

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