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MULTIPLICITIES OF THE EIGENVALUES OF THE DISCRETE SCHRÖDINGER EQUATION IN ANY DIMENSION

DAN BURGHELEA AND THOMAS KAPPELER

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ABSTRACT. The following von Neumann-Wigner type result is proved: The set of potentials $a\colon \Gamma \to \mathbf{R}$ ($\Gamma \subseteq \mathbf{Z}^N$), with the property that the corresponding discrete Schrödinger equation $\Delta_d + a$ has multiple eigenvalues when considered with certain boundary conditions, is an algebraic set of codimension≥ 2 within $\mathbf{R}^\Gamma$.

1. Introduction. The classical theorem of von Neumann and Wigner [**10**] shows that, within the space S of real symmetric $n \times n$ matrices ($n \geq 2$), the ones with multiple eigenvalue(s) form a real algebraic set of codimension 2. This implies, in particular, that the set of all real symmetric matrices with simple spectrum is pathwise connected, locally pathwise connected, and dense in S. Recently, Lax [**7**] shows that in a three-dimensional vector space of ($n \times n$) symmetric matrices there exists at least a one-dimensional subspace of matrices with multiple eigenvalues ("crossing" of eigenvalues). These results were refined and generalized by Friedland, Robbin, and Sylvester [**2**].

In this paper the subset of all real symmetric matrices is considered which come from the discrete Schrödinger equation $\Delta_d + a$ on a given subset Γ of $\mathbf{Z}^N$ and with boundary conditions to be specified later. By showing a certain transversality condition, an analogous result to the one of von Neumann and Wigner is shown: We prove that the set Q_B of all potentials a with the property that $\Delta_d + a$ has multiple eigenvalue(s) when considered with a certain boundary condition is an algebraic set of codimension ≥ 2.

To be more precise, let N and $n_1, \dots, n_N$ be arbitrary positive numbers which define a subset Γ in $\mathbf{Z}^N$ in the following way:

$$\Gamma := \{z = (z_1, \dots, z_N)\colon 1 \leq z_i \leq n_i, 1 \leq i \leq N\}.$$

For an arbitrary function $u\colon \mathbf{Z}^N \to \mathbf{R}$, let us define the discrete Laplace operator Δ_d, conveniently for our purposes, by

$$\Delta_d u(z) := \sum_{|w-z|=1} u(w) \qquad (z \in \mathbf{Z}^N),$$

where $|\cdot|$ denotes the Euclidean distance. Let $a\colon \Gamma \to \mathbf{R}$ be an arbitrary function. Then we consider the following two eigenvalue problems of the Schrödinger equation.

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I. *Dirichlet problem.*

$$\Delta_d u(z) + a(z)u(z) = \lambda u(z) \qquad (z \in \Gamma). \tag{I.1}$$

$$u(z) = 0 \quad \text{for } z \in \mathbf{Z}^N \backslash \Gamma. \tag{I.2}$$

II. *Periodic problem.*

$$\Delta_d u(z) + a(z)u(z) = \lambda u(z) \qquad (z \in \mathbf{Z}^N), \tag{II.1}$$

where a is periodically extended from Γ to the whole of $\mathbf{Z}^N$.

$$u(z_1, \ldots, z_j + n_j, \ldots, z_N) = e^{i\kappa_j} u(z_1, \ldots, z_j, \ldots, z_N) \tag{II.2}$$

for given real numbers κ_j $(1 \leq j \leq N)$.

Let us denote by Q_B the set of all potentials a in $\mathbf{R}^P$ such that $(\Delta_d + a)$, considered with the boundary condition B, has at least one multiple eigenvalue, where $P = \prod_{i=1}^N n_i$ and B is equal to I or II, staying for the different boundary conditions as given in I and II. For the reader's convenience, let us recall that by an algebraic set we mean the set of zeros of a finite collection of polynomials in some Euclidean space. An algebraic (real analytic) variety K in $\mathbf{R}^n$ is locally the locus of the zeros of a finite collection of polynomials (real analytic functions).

THEOREM. *Q_B is an algebraic set of codimension ≥ 2 in $\mathbf{R}^P$. In particular, $\mathbf{R}^P \backslash Q_B$ is connected.*

REMARK 1. The continuous analogue of the Theorem above is proved in [4].

REMARK 2. It will follow from the proof of the Theorem that the statement is true for much more general difference operators than the discrete Schrödinger operators, and more general boundary conditions than the ones given here. In particular, the Theorem will hold if we replace the Laplacian Δ_d by a quite general, not necessarily elliptic, difference operator of arbitrary order ≥ 2. Evidently the Theorem does not hold for difference operators of order zero and the Lemma in §2 fails to be true for such operators.

REMARK 3. Related papers are [**1**–**4**, **7**, **9**, **10**].

2. Proof of the Theorem. In both cases ($B =$ I or II), we can include the boundary conditions in the operator $\Delta_d + a$ which can then be represented by $P \times P$ matrices $S_{\mathrm{I}}(a)$ and $S_{\mathrm{II}}(a)$, respectively, where the potential a in $\mathbf{R}^P$ acts diagonally. For a in $\mathbf{R}^P$, we define $f_{B,a} := \det(S_B - \lambda\,\mathrm{Id})$ ($B =$I or II). $f_{B,a}$ is a polynomial in λ of degree P with coefficients which are polynomials in $a(z)$ $(z \in \Gamma)$. Clearly, $S_B(a)$ has a multiple eigenvalue iff $f_{B,a}$ has a multiple root, and this is true iff the discriminant $\mathbf{D}(f_{B,a})$ of $f_{B,a}$ does vanish (cf. e.g. [**6**, p. 60]). Because $\mathbf{D}(f_{B,a})$ is a polynomial in $a(z)$ $(z \in \Gamma)$, it follows that Q_B is an algebraic set. To compute the claimed dimension, let us fix B and introduce the following notation: For a in $\mathbf{R}^P$ given, we denote the eigenvalues of $S_B(a)$ in increasing order and with their multiplicities as follows:

$$\lambda_1(a) \leq \cdots \leq \lambda_P(a).$$

By definition, the mth eigenvalue $\lambda_m(a)$ of $S_B(a)$ has multiplicity ≥ 2 iff

$$\lambda_1(a) \leq \cdots \leq \lambda_{m-1}(a) < \lambda_m(a) = \lambda_{m+1}(a) \leq \cdots .$$

We define for $m = 1, \ldots, P-1$

$$Q_B(m) := \{a \in \mathbf{R}^P : \lambda_1(a) < \cdots < \lambda_m(a) = \lambda_{m+1}(a) \leq \cdots\},$$
$$T_B(m) := \{a \in \mathbf{R}^P : a \in Q_B(m) \text{ and } \lambda_m(a) = 0\}$$

and

$$T_B := \bigcup_{1 \leq m \leq P-1} T_B(m) \quad \text{(disjoint union)}.$$

Then $Q_B = \bigcup_{1 \leq m \leq P-1} Q_B(m)$ (disjoint union) and $Q_B(m) \cong T_B(m) \times \mathbf{R}$ by the following map:

$$T_B(m) \times \mathbf{R} \to Q_B(m), \qquad (a, c) \to a + c\mathbf{1},$$

where $\mathbf{1}$ denotes the vector in $\mathbf{R}^P$ with all entries equal to 1.

Once we have shown that $T_B(m)$ is a real analytic variety, we conclude that also $Q_B(m)$ is a real analytic variety and thus

$$\operatorname{codim} Q_B = \min_m \operatorname{codim} Q_B(m) \geq \min_m (\operatorname{codim} T_B(m)) - 1.$$

It will be shown with the Proposition below that T_B is an algebraic set of codimension ≥ 3. So we can conclude that $\operatorname{codim} T_B(m) \geq \operatorname{codim} T_B \geq 3$. In all, we get $\operatorname{codim} Q_B \geq 2$. To show that $T_B(m)$ is a real analytic variety, let a^0 be in $T_B(m)$. Following Kato [**5**], there exists a neighborhood U of a^0 in $\mathbf{R}^P$ such that for all a in U

$$\lambda_1(a) < \cdots < \lambda_m(a) \leq \lambda_{m+1}(a) \leq \cdots.$$

Then a potential a in U is an element in $T_B(m)$ iff

$$\frac{\partial^j}{\partial \lambda^j} \det(S_B(a) - \lambda \operatorname{Id}) \,|_{\lambda=0} = 0 \qquad (j = 0, 1).$$

This shows that $T_B(m)$ is a real analytic variety. □

Let us recall the definition of the sets T_B:

$$T_B := \{a \in \mathbf{R}^P : S_B(a) \text{ has 0 as a multiple eigenvalue}\}.$$

Then for $B =$I or II:

PROPOSITION. *T_B is an algebraic set of* codim ≥ 3 *in* $\mathbf{R}^P$.

PROOF. We restrict ourselves to prove the Proposition for the case where $N = 2$, $n_1 = n_2 = n$, and $B = \mathrm{I}$. It will follow from the proof that the general case is shown in the same way. We write T for T_{I}, S for S_{I}, and n^2 for P.

$S = S(a)$ then has the form of a block Jacobi matrix:

$$S(a) = \begin{pmatrix} A_1 & I & 0 & \cdots & 0 \\ I & A_2 & I & \ddots & \vdots \\ 0 & I & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & 0 & I & A_n \end{pmatrix},$$

where I denotes the $n \times n$ identity matrix and A_i are $n \times n$ Jacobi matrices ($i \leq i \leq n$) given by

$$A_i = \begin{pmatrix} a(i,1) & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & a(i,n) \end{pmatrix}.$$

A potential a in $\mathbf{R}^{n^2}$ is in T iff $\dim \operatorname{Ker} S(a) \geq 2$, where $\operatorname{Ker} S(a)$ denotes the kernel of $S(a)$. Now $\dim \operatorname{Ker} S(a) \geq 2$ iff all the $(n^2-1) \times (n^2-1)$ submatrices of $S(a)$ are singular. It follows that T is an algebraic set in $\mathbf{R}^{n^2}$ and can thus be decomposed in its irreducible components T_i, $T = \bigcup_{i=1}^{M} T_i$.

It suffices to show that $\operatorname{codim}_a T_i \geq 3$ for $1 \leq i \leq M$ and any regular point a in T_i (cf. e.g. [**8**, p. 41]). To simplify notation let T_i be T_1. Choose an arbitrary regular point a^0 in T_1. The idea of the proof is to express, in a neighborhood of a^0 in T_1, two coefficients out of a_{ik} as functions of the others, and then to show that there is a third equation among the remaining coefficients which holds on this neighborhood and which does not hold identically on $\mathbf{R}^{n^2-2}$.

Let us denote by $[\alpha, i]$ the number $(\alpha - 1)n + i$ ($1 \leq \alpha, i \leq n$). For an arbitrary $n^2 \times n^2$ matrix M we denote by $M((\alpha, i), (\beta, j))$ the $(n^2-1) \times (n^2-1)$ submatrix of M by eliminating the $[\alpha, i]$th row and the $[\beta, j]$th column. Moreover, we define $b_{[\alpha,i]} := a(\alpha, i)$.

Step 1. Let us define $F_0(a) := \det S(a)$ and $F_k(a) := \partial F_{k-1}(a)/\partial b_k$ ($1 \leq k \leq n^2$) as well as $G_k(a) := F_{k-1}(a) - b_k F_k(a)$ ($1 \leq k \leq n^2$). Clearly $F_k(a)$ and $G_k(a)$ are independent of $b_1, \ldots, b_k$. In particular, we have

$$F_{k-1}(a) = b_k F_k(a) + G_k(a) \quad \text{and} \quad \det S(a) = b_1 F_1(a) + G_1(a).$$

$F_1(a)$ vanishes identically on T. Now let us assume that $F_2(a), \ldots, F_k(a)$ are all vanishing identically in a certain neighborhood of a^0 in T_1. Due to the fact that a^0 is regular there are two possibilities:

(1) there is a neighborhood of a^0 in T_1 such that $F_2(a), \ldots, F_k(a)$ and $F_{k+1}(a)$ do vanish identically; or

(2) there exists a neighborhood of a^0 in T_1 such that F_{k+1} vanishes at most on a real analytic variety of codim $\geq \operatorname{codim} T_1 + 1$ contained in this neighborhood.

If the second possibility holds then one can solve the equation $0 = b_{k+1}F_{k+1}(a) + G_{k+1}(a)$ for b_{k+1} in a neighborhood of A^0 in T_1 except on a set of points of lower dimension.

Now

$$F_{n^2}(a) = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 \ldots & \ldots & 0 & 1 \end{pmatrix} = 1$$

and thus we conclude that there exists a smallest k, $1 \leq k \leq n^2$, such that b_k can be expressed as a real analytic function of $b_{k+1}, \ldots, b_{n^2}$ in a neighborhood V of a^0 in T_1 except on a real analytic variety of lower dimension, contained in V.

Step 2. Let us assume that in Step 1 we could express $a(\alpha, i)$ as a real analytic function of the remaining coefficients. Then consider $0 = \det S((\alpha, i), (\alpha, i))$.

Applying the same procedure for $S((\alpha, i), (\alpha, i))$ as was applied for $S(a)$ in Step 1, we conclude that there exists a neighborhood V of a^0 in T_1 and (β, j) such that $a(\beta, j)$ can be expressed as a function of $a(\gamma, k) \neq (\alpha, i)$ and $(\gamma, k) \neq (\beta, j)$ except on a set of lower dimension in V.

Step 3. Now let us assume that there exists a neighborhood V of a^0 in T_1 and $(\alpha, i), (\beta, j)$ $((\alpha, i) \neq (\beta, j))$ such that $a(\alpha, i)$ can be expressed as a real analytic function of the remaining coefficients of a, and $a(\beta, j)$ can be expressed as a real analytic function of the other coefficients different from $a(\alpha, i)$, except on a set of points of lower dimension in V. Then let us consider the equation

$$0 = \det S((\alpha, k), (\beta, j))$$

which holds on T. This is a polynomial in $a(\gamma, k)$ with $(\gamma, k) \neq (\alpha, i)$ and $\neq (\beta, j)$. It thus suffices to show that $\det S((\alpha, i), (\beta, j))$ is not identically zero on $\mathbf{R}^{n^2-2}$. This will be done with the following Lemma. □

LEMMA. $\det S((\alpha, i), (\beta, j))$ *is not identically zero on* $\mathbf{R}^{n^2-2}$.

PROOF. Clearly $\det S((\alpha, i), (\beta, j))$ is a polynomial in $a(\gamma, k)$. We have to show that

$$\deg \det S((\alpha, i), (\beta, j)) \geq 1.$$

Without loss of any generality we may and do assume that $\alpha \leq \beta$ and $i \leq j$. It then follows that $\det S((\alpha, i), (\beta, j))$ contains the following monomial:

$$\left(\prod_{\substack{1 \leq k \leq n \\ 1 \leq \gamma < \alpha \text{ or } \beta < \gamma \leq n}} a(\gamma, k) \right) * \left(\prod_{\substack{\alpha < \gamma \leq \beta \\ 1 \leq k \leq n \text{ and } k \neq j}} a(\gamma, k) \right) * \left(\prod_{\substack{1 \leq k \leq i-1 \\ j+1 \leq k \leq n}} a(\alpha, k) \right),$$

and thus the Lemma follows.

REMARK. If $j \leq i$, then $\det S((\alpha, i), (\beta, j))$ contains the following monomial:

$$\left(\prod_{\substack{1 \leq k \leq n \\ 1 \leq \gamma < \alpha \text{ or } \beta < \gamma \leq n}} a(\gamma, k) \right) * \left(\prod_{\substack{\alpha < \gamma \leq \beta \\ 1 \leq k \leq n \text{ and } k \neq j}} a(\gamma, k) \right) * \left(\prod_{\substack{1 \leq k \leq j-1 \\ j+1 \leq k \leq n}} a(\alpha, k) \right).$$

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210 (Current address of Dan Bu-ghelea)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104

Current address (T. Kappeler): Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254