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DERIVATIONS AND (HYPER)INVARIANT SUBSPACES OF A BOUNDED OPERATOR

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ABSTRACT. Let X be a complex Banach space and $\mathcal{L}(X)$ the set of bounded linear operators on X. For $T \in \mathcal{L}(X)$, a derivation Δ_T is defined by $\Delta_T A = TA - AT$ for $A \in \mathcal{L}(X)$. By induction, $\Delta_T^m = \Delta_T \circ \Delta_T^{m-1}$ is defined for each integer $m \geq 2$. We call the kernel of Δ_T^m the m-commutant of T. For a polynomially compact operator T, we consider the (hyper)invariant subspace problem for operators in the m-commutant of T for $m \geq 1$. It is easily seen that the m-commutant ($m > 1$) of T could be much larger than $\mathrm{Ker}(\Delta_T)$. So our idea is a variation of Lomonosov's theorem in [**6**]. We start with several identities on derivations, and then prove our results on the existence of (hyper)invariant subspaces. Theorem 2 in [**5**] is generalized.

In this paper, we always assume that $\dim X = \infty$. For a bounded operator T on X and a complex number α, we denote by $X_T(\alpha)$ the norm closure of the linear manifold $\bigvee_{n=1}^{\infty} \ker(T-\alpha)^n$. If A is a nonscalar bounded operator on X, we say that $\mathrm{Lat}\, A$ is nontrivial if A has a nontrivial (closed) invariant subspace. Similarly we say that H-$\mathrm{Lat}\, A$ is nontrivial if A has a nontrivial (closed) hyperinvariant subspace. For the sake of brevity, we state our main results in one theorem as follows.

THEOREM. *If $T \in \mathcal{L}(X)$ is a polynomially compact operator with minimal polynomial $p(z) = \prod_{i=1}^{k}(z-\alpha_i)^{n_i}$ ($n_i \geq 1$ for each i, $k \geq 1$; $\alpha_i \neq \alpha_j$ if $1 \leq i < j \leq k$), and if A is a nonscalar bounded operator on X which is in the $(m+1)$-commutant of T for some $m \geq 0$, then we have the following conclusions*:

(a) *if T is algebraic and $\sigma(T)$ has at least two elements, then* $\mathrm{Lat}\, A$ *is nontrivial*;

(b) *if $\sigma(p(T)) \neq \{0\}$, then H-*$\mathrm{Lat}\, A$ *is nontrivial*;

(c) *if $\sigma(p(T)) = \{0\}$, $k > 1$, and* $0 < \mathrm{rank}\, \Delta_T^m A < \infty$, *then* $\mathrm{Lat}\, A$ *is nontrivial*;

For the special cases $m \leq 1$, we have

(d) *if T is not algebraic and* $\mathrm{rank}\, \Delta_T A < \infty$, *then H-*$\mathrm{Lat}\, A$ *is nontrivial*;

(e) *if* $\mathrm{rank}\, \Delta_T A = n_0 < \infty$ *and $q(T) = [\prod_{i=1}^{k}(T-\alpha_i)]^{m_0} \neq 0$, where $m_0 = \max\{n_0, n_1, \ldots, n_k\}$, then H-*$\mathrm{Lat}\, A$ *is nontrivial.*

Because the proof of the theorem is long, we divide it into several steps. We begin with some algebraic identities.

IDENTITY I.

$$\Delta_{T^n} A^m = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} T^{n-i-1} A^{m-j-1} (\Delta_T A) A^j T^i$$

for any integer $n \geq 1$ and $m \geq 1$.

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PROOF.

$$\Delta_{T^n} A = \sum_{i=0}^{n-1} T^{n-i-1}(\Delta_T A)T^i \quad \text{(by induction)},$$

$$\Delta_{T^n} A^m = \sum_{i=0}^{n-1} T^{n-i-1}(\Delta_T A^m)T^i,$$

$$\Delta_T A^m = -\Delta_{A^m} T = -\sum_{j=0}^{m-1} A^{m-j-1}(\Delta_T A)A^j$$

$$= \sum_{j=0}^{m-1} A^{m-j-1}(\Delta_T A)A^j.$$

Hence

$$\Delta_{T^n} A^m = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} T^{n-i-1}A^{m-j-1}(\Delta_T A)A^j T^i. \quad \text{Q.E.D.}$$

IDENTITY II. $\Delta_T^n A = \sum_{i=0}^n (-1)^i \binom{n}{i} T^{n-i} A T^i$.

PROOF. By induction. Q.E.D.

IDENTITY III.

$$\Delta_{(T-\alpha)^k}^m A = \sum_{\substack{k_i+k_i'=k-1;1\le i\le m\\ 0\le k_i,k_i'\le k-1}} (T-\alpha)^{k_1+k_2+\cdots+k_m}(\Delta_T^m A)(T-\alpha)^{k_1'+\cdots+k_m'}.$$

PROOF. We use induction on m.

$$\Delta_{(T-\alpha)^k} A = \sum_{k_1+k_1'=k-1} (T-\alpha)^{k_1}(\Delta_{T-\alpha} A)(T-\alpha)^{k_1'}$$

$$= \sum_{k_1+k_1'=k-1} (T-\alpha)^{k_1}(\Delta_T A)(T-\alpha)^{k_1'}$$

by Identity I and the fact that $\Delta_{T-\alpha} = \Delta_T$ for any scalar α. Assume

$$\Delta_{(T-\alpha)^k}^{m-1} A = \sum_{\substack{k_i+k_i'=k-1;\\ 1\le i\le m-1}} (T-\alpha)^{k_1+k_2+\cdots+k_{m-1}}(\Delta_T^{m-1} A)(T-\alpha)^{k_1'+\cdots+k_{m-1}'},$$

$$\Delta_{(T-\alpha)^k}^m A = \Delta_{(T-\alpha)^k}[\Delta_{(T-\alpha)^k}^{m-1} A]$$

$$= \Delta_{(T-\alpha)^k}\left[\sum_{\substack{k_i+k_i'=k-1;\\ 1\le i\le m-1}} (T-\alpha)^{k_1+k_2+\cdots+k_{m-1}}(\Delta_T^{m-1} A)(T-\alpha)^{k_1'+\cdots+k_{m-1}'}\right]$$

$$= \sum_{\substack{k_i+k_i'=k-1;\\ 1\le i\le m-1}} (T-\alpha)^{k_1+\cdots+k_{m-1}}[\Delta_{(T-\alpha)^k}(\Delta_T^{m-1} A)](T-\alpha)^{k_1'+\cdots+k_{m-1}'}$$

$$= \sum_{\substack{k_i+k_i'=k-1\\ 1\le i\le m}} (T-\alpha)^{k_1+\cdots+k_m}(\Delta_T^m A)(T-\alpha)^{k_1'+\cdots+k_m'}$$

by the case $m=1$. Q.E.D.

IDENTITY IV.

$$\Delta^m_{(T-\alpha)^k} A = [k(T-\alpha)^{k-1}]^m (\Delta^m_T A) = (\Delta^m_T A)[k(T-\alpha)^{k-1}]^m$$

if $\Delta^{m+1}_T A = 0$ $(k, m \geq 1)$.

PROOF. $(T-\alpha)^p(\Delta^m_T A) = (\Delta^m_T A)(T-\alpha)^p$ for any integer $p \geq 0$ by the hypothesis $\Delta^{m+1}_T A = 0$. Then Identity IV follows from Identity III. Q.E.D.

IDENTITY V. Let p be any polynomial. If $\Delta^{m+1}_T A = 0$ for some integer $m \geq 1$, then

$$\Delta^m_{p(T)} A = (p'(T))^m (\Delta^m_T A) = (\Delta^m_T A)(p'(T))^m,$$

where p' is the derivative of p (Identity IV is a special case of Identity V).

PROOF. Note the following facts:

(1) If $T_1T_2 = T_2T_1$, then $\Delta_{T_1} \circ \Delta_{T_2} = \Delta_{T_2} \circ \Delta_{T_1}$ just by the definitions of Δ_{T_1} and Δ_{T_2}. Consequently $\Delta^p_{T_1} \circ \Delta^q_{T_2} = \Delta^q_{T_2} \circ \Delta^P_{T_1}$ for any integers $p, q \geq 1$.

(2) If $T_1, T_2, \ldots, T_n$ are commutative elements, then by induction,

$$\Delta^m_{\alpha_1T_1+\alpha_2T_2+\cdots+\alpha_nT_n} A = (\alpha_1\Delta_{T_1} + \alpha_2\Delta_{T_2} + \cdots + \alpha_n\Delta_{T_n})^m A$$
$$= \left[\sum_{\substack{i_i+i_2+\cdots+i_n=m \\ 0\leq i_j\leq n}} \frac{m!}{i_1!i_2!\cdots i_n!} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_n^{i_n} \Delta^{i_1}_{T_1}\Delta^{i_2}_{T_2}\cdots\Delta^{i_n}_{T_n} \right] A.$$

Let $p(T) = \alpha_1T^{k_1} + \alpha_2T^{k_2} + \cdots + \alpha_nT^{k_n}$. Then

$$\Delta^m_{p(T)} A = \left[\sum_{\substack{i_1+\cdots+i_n=m \\ 0\leq i_j\leq m}} \frac{m!}{i_1!i_2!\cdots i_n!} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_n^{i_n} \Delta^{i_1}_{T^{k_1}}\Delta^{i_2}_{T^{k_2}}\cdots\Delta^{i_n}_{T^{k_n}} \right] A$$

by facts (1) and (2). We may assume $k_i \geq 1$, $1 \leq i \leq n$, since $\Delta\beta = 0$ for any scalar β.

$$\Delta^{i_1}_{T^{k_1}}\Delta^{i_2}_{T^{k_2}}\cdots\Delta^{i_n}_{T^{k_n}} A = \Delta^{i_1}_{T^{k_1}}\cdots\Delta^{i_{n-1}}_{T^{k_{n-1}}}$$
$$\left[\sum_{j_p(n)+j'_p(n)=k_n-1;1\leq p\leq i_n} T^{j_1(n)+\cdots+j_{i_n}(n)}(\Delta^{i_n}_T A)T^{j'_1(n)+\cdots+j'_{i_n}(n)} \right]$$
$$= \sum_{\substack{j_p(n)+j'_p(n)=k_n-1 \\ 1\leq p\leq i_n}} \cdots \sum_{\substack{j_p(1)+j'_p(1)=k_1-1 \\ 1\leq p\leq i_1}} T^{r_1+\cdots+r_n}(\Delta^{i_1+\cdots+i_n}_T A)T^{r'_1+\cdots+r'_n},$$

where $r_l = \sum_{t=1}^{i_l} j_t(l)$, $r'_l = \sum_{t=1}^{i_l} j'_t(l)$. (Here we repeatedly used Identity III.) Note that

$$\sum_{l=1}^{n}(r_l + r'_l) = i_1(k_1-1) + \cdots + i_n(k_n-1),$$
$$i_1 + i_2 + \cdots + i_n = m, \; T(\Delta^m_T A) = (\Delta^m_T A)T.$$

Hence

$$\Delta^{i_1}_{T^{k_1}}\cdots\Delta^{i_n}_{T^{k_n}} A = k_1^{i_1}\cdots k_n^{i_n} T^{i_1(k_1-1)}T^{i_2(k_2-1)}\cdots T^{i_n(k_n-1)}(\Delta^m_T A)$$
$$= [k_1T^{k_1-1}]^{i_1}[k_2T^{k_2-1}]^{i_2}\cdots[k_nT^{k_n-1}]^{i_n}(\Delta^m_T A).$$

Therefore, we have

$$\Delta^m_{p(T)}A = \left[\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\le i_j\le m}} \frac{m!}{i_1!i_2!\cdots i_n!}(\alpha_1 k_1 T^{k_1-1})^{i_1}\cdots(\alpha_n k_n T^{k_n-1})^{i_n}\right](\Delta^m_T A)$$
$$= [\alpha_1 k_1 T^{k_1-1} + \alpha_2 k_2 T^{k_2-1} + \cdots + \alpha_n k_n T^{k_n-1}]^m (\Delta^m_T A)$$
$$= [p'(T)]^m(\Delta^m_T A) = (\Delta^m_T A)[p'(T)]^m. \quad \text{Q.E.D.}$$

COROLLARY. *The following three conditions are equivalent for any fixed* $m \ge 0$.
(i) $\Delta^{m+1}_T A = 0$;
(ii) $\Delta^{m+1}_{p(T)} A = 0$ *for each polynomial* p;
(iii) $\Delta^{m+1}_S A = 0$ *if* S *is in the norm closed algebra generated by* T *and* I.

PROOF. (i)⇒(ii) by Identity V.
(iii)⇒(i) is trivial.
(ii)⇒(iii) by the continuity of the operator $\Delta^{m+1}_{(\cdot)}A$ in the norm topology of $\mathcal{L}(X)$.

LEMMA. *If* $\exists T \in \mathcal{L}(X)$ *having an eigenvalue* α *such that* $X_T(\alpha) \ne X$, *and* A *is in the* $(m+1)$*-commutant of* T *for some* $m \ge 0$, *then* Lat A *is nontrivial.*

If, moreover, $\exists\alpha$ *such that either* $0 < \dim X_T(\alpha) < \infty$ *or* $0 < \dim X_{T^*}(\alpha) < \infty$, *then* H-Lat A *is nontrivial.*

PROOF. If $m = 0$, then A is in the commutant of T. This implies that $(T-\alpha)^n A = A(T-\alpha)^n$ for any integer $n \ge 1$. Hence $\ker(T-\alpha)^n$ $(n \ge 1)$ are nontrivial invariant subspaces of A by the hypothesis on α.

If $m > 0$, by Identity II,

$$\Delta^m_{(T-\alpha)^k}A = \sum_{i=1}^m (-1)^i \binom{m}{i}(T-\alpha)^{k(m-i)}A(T-\alpha)^{ki} + (T-\alpha)^{km}A$$

for any integer $k \ge 1$.

On the other hand, by Identity IV, we have $\Delta^m_{(T-\alpha)^k}A = k^m(\Delta^m_T A)(T-\alpha)^{(k-1)m}$ for any integer $k \ge 1$ since A is in the $(m+1)$-commutant of T.

Comparing the two expressions of $\Delta^m_{(T-\alpha)^k}A$, we find

$$(T-\alpha)^{km}A = k^m(\Delta^m_T A)(T-\alpha)^{(k-1)m} - \sum_{i=1}^m (-1)^i\binom{m}{i}(T-\alpha)^{k(m-i)}A(T-\alpha)^{ki}$$

for any $k \ge 1$.

If $k \ge 2$, then

$$(T-\alpha)^{km}A = \left[k^m(\Delta^m_T A)(T-\alpha)^{(k-1)(m-1)} - \sum_{i=1}^m (-1)^i\binom{m}{i}(T-\alpha)^{k(m-i)}A(T-\alpha)^{ki-k+1}\right](T-\alpha)^{k-1}.$$

Therefore, $A(\ker(T-\alpha)^{k-1}) \subset \ker(T-\alpha)^{km}$ for any $k \geq 2$; and so

$$A\left(\bigvee_{n=1}^{\infty} \ker(T-\alpha)^n\right) \subset \bigvee_{n=1}^{\infty} \ker(T-\alpha)^n.$$

The continuity of A implies that $X_T(\alpha)$ is a nontrivial invariant subspace of A. If $0 < \dim X_T(\alpha) < \infty$, then $A(X_T(\alpha)) \subset X_T(\alpha)$ implies that A has an eigenvalue λ. It follows that $\ker(A-\lambda)$ is a nontrivial hyperinvariant subspace of A since A is nonscalar.

Note

$$\begin{aligned}\Delta_{T^*}^m A^* &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} T^{*m-i} A^* T^{*i} \\ &= \left[\sum_{i=0}^{m} (-1)^i \binom{m}{i} T^i A T^{m-i}\right]^* = (-1)^m (\Delta_T^m A)^*.\end{aligned}$$

So A is in the $(m+1)$-commutant of T iff A^* is in the $(m+1)$-commutant of T^*.

If $0 < \dim X_{T^*}(\alpha) < \infty$, then $A^*(X_{T^*}(\alpha)) \subset X_{T^*}(\alpha)$ by the above argument. Hence A^* has an eigenvalue μ and the closure of the range of $A-\mu$ is a nontrivial hyperinvariant subspace of A. Q.E.D.

Now, we prove our theorem.

PROOF OF THE THEOREM. (a) Let $p_a(z) = \prod_{j=1}^{l}(z-\beta_j)^{m_j}$ be the minimal annihilating polynomial of T, where $\beta_{j_1} \neq \beta_{j_2}$ if $1 \leq j_1 < j_2 \leq l$ and $m_j \geq 1$ if $1 \leq j \leq l$.

First of all, it is easily seen that $\sigma(T) = \{\beta_1, \ldots, \beta_l\}$, $\sigma_e(T) = \{\alpha_1, \ldots, \alpha_k\}$, and $l \geq k$. If there is $\beta \in \sigma(T)\backslash\sigma_e(T)$, it is well known that $0 < \dim X_T(\beta) < \infty$. The lemma applies and H-Lat A is nontrivial. We may assume that $\sigma(T) = \sigma_e(T)$ and $\beta_i = \alpha_i$ for $1 \leq i \leq k = l$, and so $m_i \geq n_i$ for $1 \leq i \leq k$.

$p_a(T) = 0$ implies that every $\alpha_i = \beta_i$ is an eigenvalue of T. For any $1 \leq i_0 \leq k$, pick a small neighborhood U of α_{i_0} such that $U \cap (\sigma(T)\backslash\{\alpha_{i_0}\}) = \varnothing$.

Define

$$P_{i_0} = \frac{1}{2\pi i}\int_{\partial U} (z-T)^{-1}\,dz.$$

Let X_{i_0} be the corresponding subspace of X, i.e. $X_{i_0} = \overline{P_{i_0}X}$. Then X_{i_0} is a nontrivial subspace of X by the hypothesis $k \geq 2$. Moreover, $X_{i_0} = \{x \in X \mid \lim_{n\to\infty} \|(T-\alpha_{i_0})^n x\|^{1/n} = 0\}$.

Clearly, $X_T(\alpha_{i_0}) \subset X_{i_0}$. Hence $\{0\} \neq X_T(\alpha_{i_0}) \neq X$. Now the lemma applies and (a) is proved.

(b) By hypothesis, $\exists 0 \neq \alpha \in \sigma(p(T))$. It is well known that $0 < \dim X_{p(T)}(\alpha) < \infty$ since $p(T)$ is compact. Since A in the $(m+1)$-commutant of T implies A in the $(m+1)$-commutant of $p(T)$ by the corollary, the conclusion again follows from the lemma.

(c) $0 < \operatorname{rank} \Delta_T^m A < \infty$ is the hypothesis. A in the $(m+1)$-commutant of T implies that $T(\Delta_T^m A) = (\Delta_T^m A)T$. Then T has an eigenvalue α since $0 < \operatorname{rank} \Delta_T^m A < \infty$. Since $k \geq 2$, we may once more use the lemma.

(d) Since T is not algebraic, $p(T)^N \neq 0$ for any integer $N \geq 1$. If $m = 0$, then $TA = AT$, and so $p(T)A = Ap(T)$. Lomonosov's Theorem applies. If $m = 1$, then $\Delta_T^2 A = 0$. The case $\operatorname{rank} \Delta_T A = 0$ still follows from Lomonosov's Theorem. Let

$0 < \operatorname{rank} \Delta_T A < \infty$. By (b) we may also assume that $\sigma(p(T)) = \{0\}$. By the corollary of Identity V, we have $\Delta^2_{p(T)^N} A = 0$ for any integer $N \geq 1$. In particular, $p(T)(\Delta_{p(T)}A) = (\Delta_{p(T)}A)p(T)$. Note that $\operatorname{rank}(\Delta_T A) < \infty$ implies $\operatorname{rank} \Delta_{p(T)}A < \infty$ for $\Delta_{p(T)}A = p'(T)(\Delta_T A)$ by Identity V (actually $\operatorname{rank} \Delta_{p(T)}A \leq \operatorname{rank} \Delta_T A$). Let V be the range of $\Delta_{p(T)}A$, then V is a finite-dimensional invariant subspace of $p(T)$. If $V = \{0\}$, i.e. $\Delta_{p(T)}A = 0$, then Lomonosov's Theorem applies. Let $V \neq \{0\}$. Since $\sigma(p(T)) = \{0\}$ and $\sigma(p(T)|_V) \subset \eta(\sigma(p(T)) = \sigma(p(T))$ (where $\eta(\sigma(S))$ means the complement of the unbounded component of $\rho(S)$), then $p(T)|_V$ is quasinilpotent, and so $p(T)|_V$ is nilpotent since $\dim V < \infty$. Moreover, the order of $p(T)|_V$ is at most $\dim V$. Hence $[p(T)|_V]^N = 0$ if $N \geq \dim V$.

On the other hand, by Identity IV, it follows that

$$\Delta_{p(T)^{N+1}}A = (N+1)p'(T)p(T)^N(\Delta_{p(T)}A) = 0$$

if $N \geq \dim V$, i.e. $p(T)^{N+1}A = Ap(T)^{N+1}$, where $p(T)^{N+1} \neq 0$. Now, we can use Lomonosov's Theorem. (d) is proved.

(e) The case $n_0 = \operatorname{rank} \Delta_T A = 0$ is trivial by Lomonosov's Theorem. Let $1 \leq n_0 = \operatorname{rank} \Delta_T A < \infty$. Also we may assume $m = 1$. By the hypothesis, $T(\Delta_T A) = (\Delta_T A)T$. Let W be the range of $\Delta_T A$; then $W \in \operatorname{Lat} T$. $n_0 < \infty$ implies that $\sigma(T|_W)$ contains only eigenvalues, and also $\sigma(T|_W) \subset \eta(\sigma(T)) = \sigma(T)$. As in the proof of (a), we may assume $\sigma(T) = \sigma_e(T)$.

Let $\sigma(T|_W) = \{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$, $i_1 < i_2 < \cdots < i_p$, $1 \leq p \leq k$. If $p = 1$, i.e. $\sigma(T|_W)$ has only one element, then $(T|_W - \alpha_{i_1})^{n_0} = 0$. Since $m_0 \geq n_0$, $(T|_W - \alpha_{i_1})^{m_0} = 0$. Hence $\operatorname{rank}(T|_W - \alpha_{i_1})^{m_0 - 1} \leq 1$. Identity V implies

$$\begin{aligned}\Delta_{q(T)}A = q'(T)(\Delta_T A) &= \left[\sum_{i=1}^{k} m_0(T-\alpha_i)^{m_0-1} \prod_{j \neq i}(T-\alpha_j)^{m_0}\right](\Delta_T A) \\ &= m_0 \left[\prod_{j \neq i_1}(T-\alpha_j)^{m_0}\right](T-\alpha_{i_1})^{m_0-1}(\Delta_T A)\end{aligned}$$

since $(T|_W - \alpha_{i_1})^{m_0} = 0$ implies $(T - \alpha_{i_1})^{m_0}(\Delta_T A) = 0$. $\operatorname{Rank}(T|_W - \alpha_{i_1})^{m_0-1} \leq 1$ implies $\operatorname{rank}(T - \alpha_{i_1})^{m_0-1}(\Delta_T A) \leq 1$. Therefore

$$\operatorname{rank}(\Delta_{q(T)}A) = \operatorname{rank}(q(T)A - Aq(T)) \leq 1.$$

Note that $m_0 \geq n_i$ for each $1 \leq i \leq k$, and so $q(T)$ is compact. Now we can use Theorem 2 in **[4]**; H-Lat A is nontrivial.

If $1 < p \leq k$, first note that the eigenvectors for distinct eigenvalues of $T|_W$ are linearly independent. $\dim W = n_0$ implies that $(T|_W - \alpha_{i_l})^{n_0-1} = 0$ for each $1 \leq l \leq p$. Hence $(T|_W - \alpha_{i_l})^{m_0-1} = 0$, $1 \leq l \leq p$. This implies that $(T - \alpha_{i_l})^{m_0-1}(\Delta_T A) = 0$ for each $1 \leq l \leq p$.

$$\Delta_{q(T)}A = \left[\sum_{i=1}^{k} m_0(T-\alpha_i)^{m_0-1} \prod_{j \neq i}(T-\alpha_j)^{m_0}\right](\Delta_T A) = 0$$

since each summand in the above sum has a factor $\prod_{l=1}^{p}(T-\alpha_{i_l})^{m_0-1}$ which kills $(\Delta_T A)$. Hence $\Delta_{q(T)}A = q(T)A - Aq(T) = 0$. Since $q(T) \neq 0$ is compact, the conclusion holds by Lomonosov's Theorem. (e) is proved. Q.E.D.

REMARKS AND QUESTIONS. 1. In conclusion (a), $k > 1$ is essential. In [**5**], it was pointed out that there always is a nilpotent element T of order 2 such that A is in the 2-commutant of T.

2. Concerning (c), we have the following question: Can we remove the hypothesis $\operatorname{rank} \Delta_T^m A < \infty$? If $\ker(p(T)) \neq \{0\}$, then T has an eigenvalue. As in the proof of (a), we can prove that $\operatorname{Lat} A$ is nontrivial without the assumption $\operatorname{rank} \Delta_T^m A < \infty$. What about the case that T is not algebraic and $\ker(p(T)) = \{0\}$? For this case, can we expect that H-$\operatorname{Lat} A$ is nontrivial? Also if we keep the condition $\operatorname{rank} \Delta_T^m A < \infty$, can we remove the condition $k > 1$? For the case $m \leq 1$, (d) says "yes."

3. If $\exists m \geq 0$ and a nonzero compact operator T such that $\Delta_T^m A$ is a rank one operator, what can we say about the (hyper)invariant subspace of A? For $m = 0$, see [**4**].

4. Combining (b) and (e) above we obtain an improvement of Theorem 2 in [**5**].

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