**DERIVATIONS AND (HYPER)INVARIANT SUBSPACES OF A BOUNDED OPERATOR**

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**Abstract.** Let $X$ be a complex Banach space and $\mathcal{L}(X)$ the set of bounded linear operators on $X$. For $T \in \mathcal{L}(X)$, a derivation $\Delta_T$ is defined by $\Delta_T A = TA - AT$ for $A \in \mathcal{L}(X)$. By induction, $\Delta_T^m = \Delta_T \circ \Delta_T^{m-1}$ is defined for each integer $m \geq 2$. We call the kernel of $\Delta_T^m$ the $m$-commutant of $T$. For a polynomially compact operator $T$, we consider the (hyper)invariant subspace problem for operators in the $m$-commutant of $T$ for $m \geq 1$. It is easily seen that the $m$-commutant $(m > 1)$ of $T$ could be much larger than $\operatorname{Ker}(\Delta_T)$. So our idea is a variation of Lomonosov's theorem in [6]. We start with several identities on derivations, and then prove our results on the existence of (hyper)invariant subspaces. Theorem 2 in [5] is generalized.

In this paper, we always assume that $\dim X = \infty$. For a bounded operator $T$ on $X$ and a complex number $\alpha$, we denote by $X_T(\alpha)$ the norm closure of the linear manifold $\bigvee_{n=1}^{\infty} \ker(T - \alpha)^n$. If $A$ is a nonscalar bounded operator on $X$, we say that $\operatorname{Lat} A$ is nontrivial if $A$ has a nontrivial (closed) invariant subspace. Similarly we say that $H-\operatorname{Lat} A$ is nontrivial if $A$ has a nontrivial (closed) hyperinvariant subspace. For the sake of brevity, we state our main results in one theorem as follows.

**Theorem.** If $T \in \mathcal{L}(X)$ is a polynomially compact operator with minimal polynomial $p(z) = \prod_{i=1}^{k} (z - \alpha_i)^{n_i}$ ($n_i \geq 1$ for each $i$, $k \geq 1$; $\alpha_i \neq \alpha_j$ if $1 \leq i < j \leq k$), and if $A$ is a nonscalar bounded operator on $X$ which is in the $(m+1)$-commutant of $T$ for some $m \geq 0$, then we have the following conclusions:

1. If $T$ is algebraic and $\sigma(T)$ has at least two elements, then $\operatorname{Lat} A$ is nontrivial;
2. If $\sigma(p(T)) \neq \{0\}$, then $H-\operatorname{Lat} A$ is nontrivial;
3. If $\sigma(p(T)) = \{0\}$, $k > 1$, and $0 < \operatorname{rank} \Delta_T^m A < \infty$, then $\operatorname{Lat} A$ is nontrivial;

For the special cases $m \leq 1$, we have

1. If $T$ is not algebraic and rank $\Delta_T A < \infty$, then $H-\operatorname{Lat} A$ is nontrivial;
2. If rank $\Delta_T A = n_0 < \infty$ and $q(T) = [\prod_{i=1}^{k} (T - \alpha_i)]^{m_0} \neq 0$, where $m_0 = \max\{n_0, n_1, \ldots, n_k\}$, then $H-\operatorname{Lat} A$ is nontrivial.

Because the proof of the theorem is long, we divide it into several steps. We begin with some algebraic identities.

**Identity I.**

$$\Delta_T^n A^m = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} T^{n-i-1} A^{m-j-1} (\Delta_T A) A^j T^i$$

for any integer $n \geq 1$ and $m \geq 1$.
Proof.

\[ \Delta_T A^n = \sum_{i=0}^{n-1} T^{n-i-1}(\Delta_T A)T^i \] (by induction),

\[ \Delta_T A^m = \sum_{i=0}^{n-1} T^{n-i-1}(\Delta_T A^m)T^i, \]

\[ \Delta_T A^m = -\Delta A^m T = -\sum_{j=0}^{m-1} A^{m-j-1}(\Delta_T A)A^j \]

\[ = \sum_{j=0}^{m-1} A^{m-j-1}(\Delta_T A)A^j. \]

Hence

\[ \Delta_T A^m = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} T^{n-i-1}A^{m-j-1}(\Delta_T A)A^jT^i. \] Q.E.D.

Identity II. \( \Delta_T^n A = \sum_{i=0}^{n} (-1)^i \binom{n}{i} T^{n-i} A T^i. \)

Proof. By induction. Q.E.D.

Identity III.

\[ \Delta_{(T-\alpha)k} A = \sum_{k_i + k'_i = k-1; 0 \leq k_i, k'_i \leq k-1} \sum_{i=0}^{m-1} (T - \alpha)^{k_1 + k_2 + \cdots + k_m} (\Delta_T^m A)(T - \alpha)^{k_1' + \cdots + k'_m}. \]

Proof. We use induction on \( m \).

\[ \Delta_{(T-\alpha)k} A = \sum_{k_1 + k'_1 = k-1} (T - \alpha)^{k_1} (\Delta_{T-\alpha} A)(T - \alpha)^{k'_1} \]

\[ = \sum_{k_1 + k'_1 = k-1} (T - \alpha)^{k_1} (\Delta_T A)(T - \alpha)^{k'_1} \]

by Identity I and the fact that \( \Delta_{T-\alpha} = \Delta_T \) for any scalar \( \alpha \). Assume

\[ \Delta_{m-1}^{m-1} (T-\alpha)k A = \sum_{k_i + k'_i = k-1; 1 \leq i \leq m-1} (T - \alpha)^{k_1 + k_2 + \cdots + k_{m-1}} (\Delta_T^{m-1} A)(T - \alpha)^{k_1' + \cdots + k'_{m-1}}, \]

\[ \Delta_m^{m-1} (T-\alpha)k A = \Delta(T-\alpha)k [\Delta_{(T-\alpha)k} A] \]

\[ = \Delta(T-\alpha)k \left[ \sum_{k_i + k'_i = k-1; 1 \leq i \leq m-1} (T - \alpha)^{k_1 + k_2 + \cdots + k_{m-1}} (\Delta_T^{m-1} A)(T - \alpha)^{k_1' + \cdots + k'_{m-1}} \right] \]

\[ = \sum_{k_i + k'_i = k-1; 1 \leq i \leq m-1} (T - \alpha)^{k_1 + \cdots + k_m} [\Delta(T-\alpha)k (\Delta_T^{m-1} A)](T - \alpha)^{k_1' + \cdots + k'_m} \]

\[ = \sum_{k_i + k'_i = k-1; 1 \leq i \leq m} (T - \alpha)^{k_1 + \cdots + k_m} (\Delta_T^m A)(T - \alpha)^{k_1' + \cdots + k'_m} \]

by the case \( m = 1 \). Q.E.D.
IDENTITY IV.

\[ \Delta_{(T-\alpha)}^m A = [k(T-\alpha)^{k-1}]^m (\Delta_T^m A) = (\Delta_T^m A)[k(T-\alpha)^{k-1}]^m \]

if \( \Delta_T^{m+1} A = 0 \) \((k, m \geq 1)\).

**PROOF.** \( (T-\alpha)^p(\Delta_T^m A) = (\Delta_T^m A)(T-\alpha)^p \) for any integer \( p \geq 0 \) by the hypothesis \( \Delta_T^{m+1} A = 0 \). Then Identity IV follows from Identity III. Q.E.D.

IDENTITY V. Let \( p \) be any polynomial. If \( \Delta_T^{m+1} A = 0 \) for some integer \( m \geq 1 \), then

\[ \Delta_{p(T)}^m A = (p'(T))^m (\Delta_T^m A) = (\Delta_T^m A)(p'(T))^m, \]

where \( p' \) is the derivative of \( p \) (Identity IV is a special case of Identity V).

**PROOF.** Note the following facts:

1. If \( T_1, T_2 \) are commutative elements, then by induction,

\[ \Delta_{\alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_n T_n}^m A = (\alpha_1 \Delta_{T_1} + \alpha_2 \Delta_{T_2} + \cdots + \alpha_n \Delta_{T_n})^m A \]

Let \( p(T) = \alpha_1 T^{k_1} + \alpha_2 T^{k_2} + \cdots + \alpha_n T^{k_n} \). Then

\[ \Delta_{p(T)}^m A = \left[ \sum_{0 \leq i_1 \leq m} \frac{m!}{i_1! i_2! \cdots i_n!} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_n^{i_n} \Delta_{T_1}^{i_1} \Delta_{T_2}^{i_2} \cdots \Delta_{T_n}^{i_n} \right] A \]

by facts (1) and (2). We may assume \( k_i \geq 1, 1 \leq i \leq n, \) since \( \Delta \beta = 0 \) for any scalar \( \beta \).

\[ \Delta_{T^{k_1}}^{i_1} \Delta_{T^{k_2}}^{i_2} \cdots \Delta_{T^{k_n}}^{i_n} A = \Delta_{T^{k_1}}^{i_1} \cdots \Delta_{T^{k_n}}^{i_n-1} \]

\[ = \left[ \sum_{j_p(n)+j'_p(n)=k_n-1} \sum_{1 \leq p \leq i_n} T_j^{i_1(n)} + \cdots + \sum_{1 \leq p \leq i_1} T_{r_1}^{i_1(n)} \right] \]

\[ \Delta_{T^{r_1}}^{i_1} \cdots \Delta_{T^{r_n}}^{i_n} A \]

where \( r_1 = \sum_{l=1}^{i_1} j_l(l), r'_1 = \sum_{l=1}^{i_1} j'_l(l). \) (Here we repeatedly used Identity III.)

Note that

\[ \sum_{l=1}^{n} (r_l + r'_l) = i_1 (k_1 - 1) + \cdots + i_n (k_n - 1), \]

\[ i_1 + i_2 + \cdots + i_n = m, \]

\[ T(\Delta_T^m A) = (\Delta_T^m A)T. \]

Hence

\[ \Delta_{T^{k_1}}^{i_1} \cdots \Delta_{T^{k_n}}^{i_n} A = \left[ k_1 T^{k_1-1} \right]^{i_1} \left[ k_2 T^{k_2-1} \right]^{i_2} \cdots \left[ k_n T^{k_n-1} \right]^{i_n} (\Delta_T^m A). \]
Therefore, we have

\[
\Delta^m_{p(T)} A = \left[ \sum_{0 \leq i_1 + \cdots + i_n = m} \frac{m!}{i_1! \cdots i_n!} (\alpha_1 k_1 T^{k_1-1})^{i_1} \cdots (\alpha_n k_n T^{k_n-1})^{i_n} \right] (\Delta^m_T A)
\]

\[
= [\alpha_1 k_1 T^{k_1-1} + \alpha_2 k_2 T^{k_2-1} + \cdots + \alpha_n k_n T^{k_n-1}]^m (\Delta^m_T A)
\]

\[
= [p'(T)]^m (\Delta^m_T A) = (\Delta^m_T A)[p'(T)]^m. \quad \text{Q.E.D.}
\]

**Corollary.** The following three conditions are equivalent for any fixed \( m \geq 0 \).

(i) \( \Delta^{m+1}_T A = 0 \);

(ii) \( \Delta^m_{p(T)} A = 0 \) for each polynomial \( p \);

(iii) \( \Delta^m_{S+1} A = 0 \) if \( S \) is in the norm closed algebra generated by \( T \) and \( I \).

**Proof.** (i) \( \Rightarrow \) (ii) by Identity V.

(ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i) by the continuity of the operator \( \Delta^{m+1}_T A \) in the norm topology of \( \mathcal{L}(X) \).

**Lemma.** If \( \exists T \in \mathcal{L}(X) \) having an eigenvalue \( \alpha \) such that \( X_T(\alpha) \neq X \), and \( A \) is in the \((m+1)\)-commutant of \( T \) for some \( m \geq 0 \), then \( \text{Lat} \ A \) is nontrivial.

If, moreover, \( \exists \alpha \) such that either \( 0 < \dim X_T(\alpha) < \infty \) or \( 0 < \dim X_{T^*}(\alpha) < \infty \), then \( H \cdot \text{Lat} \ A \) is nontrivial.

**Proof.** If \( m = 0 \), then \( A \) is in the commutant of \( T \). This implies that \( (T - \alpha)^n A = A(T - \alpha)^n \) for any integer \( n \geq 1 \). Hence \( \ker (T - \alpha)^n \) \( (n \geq 1) \) are nontrivial invariant subspaces of \( A \) by the hypothesis on \( \alpha \).

If \( m > 0 \), by Identity II,

\[
\Delta^m_{(T - \alpha)^k} A = \sum_{i=1}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} (T - \alpha)^{k(m-i)} A(T - \alpha)^{ki} (T - \alpha)^{km} A
\]

for any integer \( k \geq 1 \).

On the other hand, by Identity IV, we have \( \Delta^m_{(T - \alpha)^k} A = k^m (\Delta^m_T A)(T - \alpha)^{(k-1)m} \) for any integer \( k \geq 1 \) since \( A \) is in the \((m+1)\)-commutant of \( T \).

Comparing the two expressions of \( \Delta^m_{(T - \alpha)^k} A \), we find

\[
(T - \alpha)^{km} A = k^m (\Delta^m_T A)(T - \alpha)^{(k-1)m} - \sum_{i=1}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} (T - \alpha)^{k(m-i)} A(T - \alpha)^{ki}
\]

for any \( k \geq 1 \).

If \( k \geq 2 \), then

\[
(T - \alpha)^{km} A = \left[ k^m (\Delta^m_T A)(T - \alpha)^{(k-1)(m-1)} \right.
\]

\[
- \sum_{i=1}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} (T - \alpha)^{k(m-i)} A(T - \alpha)^{ki-k+1} \left. \right] (T - \alpha)^{k-1}.
\]
Therefore, \( A(\ker(T - \alpha)^{k-1}) \subset \ker(T - \alpha)^{km} \) for any \( k \geq 2 \); and so
\[
A \left( \bigvee_{n=1}^{\infty} \ker(T - \alpha)^n \right) \subset \bigvee_{n=1}^{\infty} \ker(T - \alpha)^n.
\]

The continuity of \( A \) implies that \( X_T(\alpha) \) is a nontrivial invariant subspace of \( A \). If \( 0 < \dim X_T(\alpha) < \infty \), then \( A(X_T(\alpha)) \subset X_T(\alpha) \) implies that \( A \) has an eigenvalue \( \lambda \). It follows that \( \ker(A - \lambda) \) is a nontrivial hyperinvariant subspace of \( A \) since \( A \) is nonscalar.

Note
\[
\Delta_T^m A^* = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left( T^* - A^* \right)^{m-i} = (-1)^m (\Delta_T^m A)^*.
\]

So \( A \) is in the \((m + 1)\)-commutant of \( T \) iff \( A^* \) is in the \((m + 1)\)-commutant of \( T^* \).

If \( 0 < \dim X_T^*(\alpha) < \infty \), then \( A^*(X_T^*(\alpha)) \subset X_T^*(\alpha) \) by the above argument. Hence \( A^* \) has an eigenvalue \( \mu \) and the closure of the range of \( A - \mu \) is a nontrivial hyperinvariant subspace of \( A \). Q.E.D.

Now, we prove our theorem.

PROOF OF THE THEOREM. (a) Let \( p_a(z) = \prod_{j=1}^{l} (z - \beta_j)^{m_j} \) be the minimal annihilating polynomial of \( T \), where \( \beta_j \neq \beta_j \) if \( 1 \leq j_1 < j_2 \leq l \) and \( m_j \geq 1 \) if \( 1 \leq j \leq l \).

First of all, it is easily seen that \( \sigma(T) = \{\beta_1, \ldots, \beta_l\} \), \( \sigma_e(T) = \{\alpha_1, \ldots, \alpha_k\} \), and \( l \geq k \). If there is \( \beta \in \sigma(T) \setminus \sigma_e(T) \), it is well known that \( 0 < \dim X_T(\beta) < \infty \). The lemma applies and \( H-Lat A \) is nontrivial. We may assume that \( \sigma(T) = \sigma_e(T) \) and \( \beta_i = \alpha_i \) for \( 1 \leq i \leq k = l \), and so \( m_i \geq n_i \) for \( 1 \leq i \leq k \).

\( p_a(T) = 0 \) implies that every \( \alpha_i = \beta_i \) is an eigenvalue of \( T \). For any \( 1 \leq i_0 \leq k \), pick a small neighborhood \( U \) of \( \alpha_{i_0} \) such that \( U \cap (\sigma(T) \setminus \{\alpha_{i_0}\}) = \emptyset \).

Define
\[
P_{i_0} = \frac{1}{2\pi i} \int_{\partial U} (z - T)^{-1} \, dz.
\]

Let \( X_{i_0} \) be the corresponding subspace of \( X \), i.e. \( X_{i_0} = \overline{P_{i_0}X} \). Then \( X_{i_0} \) is a nontrivial subspace of \( X \) by the hypothesis \( k \geq 2 \). Moreover, \( X_{i_0} = \{x \in X | \lim_{n \to \infty} \|(T - \alpha_{i_0})^n x\|^{1/n} = 0\} \).

Clearly, \( X_T(\alpha_{i_0}) \subset X_{i_0} \). Hence \( \{0\} \neq X_T(\alpha_{i_0}) \neq X \). Now the lemma applies and (a) is proved.

(b) By hypothesis, \( \exists 0 \neq \alpha \in \sigma(p(T)) \). It is well known that \( 0 < \dim X_{p(T)}(\alpha) < \infty \) since \( p(T) \) is compact. Since \( A \) in the \((m + 1)\)-commutant of \( T \) implies \( A \) in the \((m + 1)\)-commutant of \( p(T) \) by the corollary, the conclusion again follows from the lemma.

(c) \( 0 < \text{rank } \Delta_T^m A < \infty \) is the hypothesis. \( A \) in the \((m + 1)\)-commutant of \( T \) implies that \( T(\Delta_T^m A) = (\Delta_T^m A)T \). Then \( T \) has an eigenvalue \( \alpha \) since \( 0 < \text{rank } \Delta_T^m A < \infty \). Since \( k \geq 2 \), we may once more use the lemma.

(d) Since \( T \) is not algebraic, \( p(T)^N \neq 0 \) for any integer \( N \geq 1 \). If \( m = 0 \), then \( TA = AT \), and so \( p(T)A = Ap(T) \). Lomonosov's Theorem applies. If \( m = 1 \), then \( \Delta_T^2 A = 0 \). The case \( \text{rank } \Delta_T A = 0 \) still follows from Lomonosov's Theorem. Let
0 < \text{rank} \Delta_T A < \infty. \ By \ (b) \ \text{we may also assume that } \sigma(p(T)) = \{0\}. \ By \ the \ corollary \ of \ \text{Identity} \ V, \ \text{we have } \Delta_{p(T)}^N A = 0 \ for \ any \ integer \ N \geq 1. \ In \ particular, \ \text{p}(T)(\Delta_{p(T)} A) = (\Delta_{p(T)} A)p(T). \ Note \ \text{that rank}(\Delta_T A) < \infty \ implies \ \text{rank} \Delta_{p(T)} A < \infty \ for \ \text{p}(T)(\Delta_{p(T)} A) = p'(T)(\Delta_{p(T)} A) \ \text{by Identity} \ V. \ \text{While} \ \text{rank} \Delta_{p(T)} A \leq \text{rank} \Delta_T A.

Let \ V \ be \ the \ range \ of \ \Delta_{p(T)} A, \ \text{then} \ V \ \text{is a finite-dimensional invariant subspace of } p(T). \ 
If \ V = \{0\}, \ i.e. \ \Delta_{p(T)} A = 0, \ \text{then} \ \text{Lomonosov's Theorem applies.} \ 
Let \ V \neq \{0\}. \ \text{Since } \sigma(p(T)) = \{0\} \ \text{and } \sigma(p(T)|_V) \subset \eta(\sigma(p(T))) = \sigma(p(T)) \ \text{(where } \eta(\sigma(S)) \ \text{means the complement of the unbounded component of } \rho(S)), \ \text{then } p(T)|_V \ \text{is quasinilpotent, and so } p(T)|_V \ \text{is nilpotent since } \text{dim} V < \infty. \ \text{Moreover, the order of } p(T)|_V \ \text{is at most } \text{dim} V. \ \text{Hence } [p(T)|_V]^N = 0 \ if \ N \geq \text{dim} V.

On the other hand, by \ \text{Identity} IV, \ \text{it follows that}

$$\Delta_{p(T)}^{N+1} A = (N + 1)p'(T)p(T)^N(\Delta_{p(T)} A) = 0$$

if \ N \geq \text{dim} V, \ i.e. \ \text{p}(T)^{N+1} A = A\text{p}(T)^{N+1}, \ \text{where } \text{p}(T)^{N+1} \neq 0. \ \text{Now, we can use Lomonosov's Theorem.} \ \text{(d) is proved.}

(e) \ \text{The case } n_0 = \text{rank} \Delta_T A = 0 \ \text{is trivial by Lomonosov's Theorem.} \ \text{Let} \ 1 \leq n_0 = \text{rank} \Delta_T A < \infty. \ \text{Also we may assume } m = 1. \ \text{By the hypothesis,} \ T(\Delta_T A) = (\Delta_T A)T. \ \text{Let } W \ \text{be the range of } \Delta_T A; \ \text{then } W \in \text{Lat} T. \ \text{n}_0 < \infty \ \text{implies that} \ \sigma(T|_W) \ \text{contains only eigenvalues, and also } \sigma(T|_W) \subset \eta(\sigma(T)) = \sigma(T).

As \ in \ the \ proof \ of \ (a), \ \text{we may assume } \sigma(T) = \sigma_e(T).

Let \ \sigma(T|_W) = \{\alpha_i, \ldots, \alpha_p\}, \ i_1 < i_2 < \cdots < i_p, \ 1 \leq p \leq k. \ \text{If } p = 1, \ i.e. \ \sigma(T|_W) \ \text{has only one element, then } (T|_W - \alpha_{i_1})^n_0 = 0. \ \text{Since } m_0 \geq n_0, \ (T|_W - \alpha_{i_1})^m_0 = 0. \ \text{Hence } \text{rank}(T|_W - \alpha_{i_1})^m_0 - 1 \leq 1. \ \text{Identity} V \implies

$$\Delta_q(T) A = q'(T)(\Delta_T A) = \left[ \sum_{i=1}^k m_0(T - \alpha_i)^m_0 - 1 \prod_{j \neq i}(T - \alpha_j)^m_0 \right] (\Delta_T A)$$

since \ (T|_W - \alpha_{i_1})^m_0 = 0 \ implies \ (T - \alpha_{i_1})^m_0 (\Delta_T A) = 0. \ \text{Rank}(T|_W - \alpha_{i_1})^m_0 - 1 \leq 1 \ \text{implies} \ \text{rank}(T - \alpha_{i_1})^m_0 - 1(\Delta_T A) \leq 1. \ \text{Therefore}

$$\text{rank}(\Delta_q(T) A) = \text{rank}(q(T) A - A_q(T)) \leq 1.$$  

Note that \ m_0 \geq n_i \ for each \ 1 \leq i \leq k, \ and \ so \ q(T) \ is compact. \ \text{Now we can use Theorem 2 in [4]; } H-\text{Lat} A \ \text{is nontrivial.}

If \ 1 < p \leq k, \ first \ note \ that \ the \ eigenvectors \ for \ distinct \ eigenvalues \ of \ T|_W \ \text{are linearly independent}. \ \text{dim} W = n_0 \ implies \ that \ (T|_W - \alpha_{i_l})^m_0 - 1 = 0 \ for \ each \ 1 \leq l \leq p. \ \text{Hence } (T|_W - \alpha_{i_l})^m_0 - 1 = 0, \ 1 \leq l \leq p. \ \text{This implies that} \ (T - \alpha_{i_l})^m_0 - 1(\Delta_T A) = 0 \ \text{for each } 1 \leq l \leq p.

$$\Delta_q(T) A = \sum_{i=1}^k m_0(T - \alpha_i)^m_0 - 1 \prod_{j \neq i}(T - \alpha_j)^m_0 \left[ (\Delta_T A) = 0 \right.$$  

since \ each \ summand \ in \ the \ above \ sum \ has \ a \ factor \ \prod_{l=1}^k (T - \alpha_{i_l})^m_0 - 1 \ which \ kills \ \text{rank}(\Delta_T A). \ \text{Hence} \ \Delta_q(T) A = q(T) A - A_q(T) = 0. \ \text{Since} \ q(T) \neq 0 \ \text{is compact, the conclusion holds by Lomonosov's Theorem. (e) is proved.} \ \text{Q.E.D.}
Remarks and Questions. 1. In conclusion (a), \( k > 1 \) is essential. In [5], it was pointed out that there always is a nilpotent element \( T \) of order 2 such that \( A \) is in the 2-commutant of \( T \).

2. Concerning (c), we have the following question: Can we remove the hypothesis \( \text{rank } \Delta^m_T A < \infty \)? If \( \ker(p(T)) \neq \{0\} \), then \( T \) has an eigenvalue. As in the proof of (a), we can prove that \( \text{Lat } A \) is nontrivial without the assumption \( \text{rank } \Delta^m_T A < \infty \). What about the case that \( T \) is not algebraic and \( \ker(p(T)) = \{0\} \)? For this case, can we expect that \( H-\text{Lat } A \) is nontrivial? Also if we keep the condition \( \text{rank } \Delta^m_T A < \infty \), can we remove the condition \( k > 1 \)? For the case \( m \leq 1 \), (d) says "yes."

3. If \( \exists m \geq 0 \) and a nonzero compact operator \( T \) such that \( \Delta^m_T A \) is a rank one operator, what can we say about the (hyper)invariant subspace of \( A \)? For \( m = 0 \), see [4].

4. Combining (b) and (e) above we obtain an improvement of Theorem 2 in [5].

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