EXTENSIONS OF CERTAIN COMPACT OPERATORS ON VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. For any compact Hausdorff spaces $X, Y$ with $\varphi: X \rightarrow Y$ a continuous onto mapping, $E, F$, Hausdorff locally convex spaces with $F$ complete, $C(X,E)$ ($C(Y,E)$) all $E$-valued continuous functions on $X$ ($Y$), and $L: C(Y,E) \rightarrow F$ a $T$-compact continuous operator ($\sigma(F,F') \leq T \leq \tau(F,F')$), it is proved there exists a $T$-compact continuous operator $L_0: C(X,E) \rightarrow F$ such that $L_0(f \circ \varphi) = L(f)$ for every $f \in C(Y,E)$.

In this paper $X, Y$ are compact Hausdorff spaces, $\varphi: X \rightarrow Y$ a continuous onto function, $E, F$ Hausdorff locally convex spaces over $K$, the field of real or complex numbers, and $C(X,E)$ ($C(Y,E)$) all $E$-valued continuous functions on $X$ (resp. $Y$). The space $C(Y,E) \circ \varphi$ is a subspace of $C(X,E)$. When $E, F$ are Banach spaces it is proved in [1, 2] that every weakly compact operator $T: C(Y,E) \rightarrow F$ has extension to a weakly compact operator $T_0: C(X,E) \rightarrow F$ in the sense that $T_0(f \circ \varphi) = T(f)$ for every $f \in C(Y,E)$. Here we will prove the result for general locally convex spaces $E, F$ assuming $F$ to be complete, by using the measure extension techniques discussed in [8]. On $C(X,E)$ or $C(Y,E)$, $u$ will denote the uniform topology. For locally convex spaces $G_1, G_2$, an operator $T: G_1 \rightarrow G_2$ will be called compact if bounded sets of $G_1$ are mapped into relatively compact subsets of $G_2$. For locally convex spaces we refer to [9]. $\mathcal{L}(E,F)$ will denote the space of all continuous linear operators from $E$ to $F$. Let $\{| \cdot |_p: p \in P\}$ be the family of all continuous seminorms on $E$. $M(X), M(Y)$ will denote all regular scalar Borel measures on $X$ and $Y$ resp. If $T: (C(Y,E), u) \rightarrow F$ is continuous and $f \in F'$, then $f \circ T \in (C(Y,E), u)'$ and so [5, 7] there exists $p \in P$ such that $|f \circ T|_p \in M^+(Y)$ (note for $g \in C(Y), g \geq 0$, $|(f \circ T)|_p(g) = \sup\{|(f \circ T)(h)|: h \in C(Y,E) \text{ and } \|h\|_p \leq f\}$, where $\|h\|_p(y) = \|h(y)\|_p$ [7]). Here $C(Y)$ stands for all $K$-valued continuous functions on $Y$. Also $(C(Y,E), u)' = M(Y,E')$ [5, 7]. $\mathcal{B}(X), \mathcal{B}(Y)$ will denote all Borel subsets of $X$ and $Y$ respectively. For an algebra $\mathfrak{A}$ of subsets of a set $Z$, $S(\mathfrak{A})$ will denote all $K$-valued $\mathfrak{A}$-simple functions on $Z$.

THEOREM. Assume $F$ is a complete locally convex space, and let $\tau$ be another locally convex topology on $F$ such that $\sigma(F,F') \leq \tau \leq \tau(F,F')$. Let $L: (C(Y,E), u) \rightarrow F$ be a continuous $\tau$-compact operator, i.e., bounded subsets of $C(Y,E)$ are mapped into relatively $\tau$-compact subsets of $F$. Then there exists a continuous...
\textbf{\textit{T}-compact operator} \( \mathbf{L}_0: (C(X, E), \mu) \rightarrow F \) such that \( \mathbf{L}_0(f \circ \varphi) = \mathbf{L}(f) \) for each \( f \in C(Y, E) \).

\textbf{Proof.} \( \mathcal{L}(E, F) \) is the space of all continuous linear operators from \( E \) to \( F \). Let \( \mathcal{F} \) be the weak completion of \( F \) and \( G \) the space of all continuous linear operators from \( E \) into \((\mathcal{F}, \sigma(\mathcal{F}, F'))\). For any finite subset \( H \subset F' \) and any bounded \( B \subset E \), a seminorm \( m \) is generated on \( G \):

\[ m(Q) = \sup\{|f \circ Q(x)| : x \in B, f \in H\}, \]

\( Q \in G \). Under the locally convex topology generated by these seminorms, \( G \) is a complete locally convex space. The topology on \( \mathcal{L}(E, F) \) is the one induced by \( G \). Since \( L \) is weakly compact, we get \([3, 5]\) a regular Borel measure \( \mu_1: \mathcal{B}(Y) \rightarrow \mathcal{L}(E, F) \) with the properties:

(I) For any \( f \in F' \), \( f \circ \mu_1 = f \circ L \).

(II) For any equicontinuous set \( H \subset F' \), there exists a \( p \in P \) such that

\[ \sup \left\{ \left| \sum f(\mu_1(A_i)(x_i)) \right| \right\} < \infty, \]

where the supremum is taken over \( f \in H \), all finite Borel partitions \( \{A_i\} \) of \( Y \), and all \( x_i \in E \) satisfying \( |x_i|_p \leq 1 \).

(III) For any bounded set \( B \subset E \), \( \left\{ \sum \mu_1(A_i)x_i \right\} \), where \( \{A_i\} \) varies over all finite disjoint collections of Borel subsets of \( X \) and \( x_i \in B \), is relatively \( \tau \)-compact in \( F \).

Let \( \{\gamma_s : s \in S\} \) be the family of all continuous seminorms on \( G \). In the notation of \([8, \text{p. 160}]\), let \( \mathcal{U} = \{\varphi^{-1}(A) : A \in \mathcal{B}(Y)\} \); defines \( \mu: \mathcal{U} \rightarrow \mathcal{L}(E, F) \), \( \mu(\varphi^{-1}(A)) = \mu_1(A), A \in \mathcal{B}(Y) \). Each \( s \in S \) gives an exhaustive, order \( \sigma \)-continuous submeasure \( \mu_s: \mathcal{U} \rightarrow [0, \infty) \), \( \mu_s(B) = \sup\{\mu(A)|_{|s} : A \in \mathcal{U}, A \subset B\} \) for every \( B \in \mathcal{U} \). As in the proof of \([8, \text{Theorem 1, pp. 160–162}]\), these submeasures can be extended to exhaustive, order \( \sigma \)-continuous, regular submeasures \( \mu_s: \mathcal{B}(X) \rightarrow [0, \infty) \) with the properties

(i) for any \( s(1), s(2) \in S \), \( \mu_{s(1)} \leq \mu_{s(2)} \) implies \( \mu_{s(1)} = \mu_{s(2)} \),

(ii) for \( \varepsilon > 0, s \in S \) and \( B \in \mathcal{B}(X) \), there exists \( B_0 \in \mathcal{U} \), such that \( \mu_s(B \Delta B_0) < \varepsilon \) (here \( B \Delta B_0 = (B \setminus B_0) \cup (B_0 \setminus B) \)).

On \( \mathcal{B}(X) \) we define \( \mathcal{F}-N \) topology \( \mathcal{T} \) generated by \( \{\mu_s : s \in S\} \) \([4, \text{p. 271}]\). \( \mathcal{B}(X) \) becomes a topological ring in which \( \mathcal{U} \) is dense. This means the uniformly continuous mapping \( \mu: \mathcal{U} \rightarrow G \) can be uniquely extended to a uniformly continuous mapping \( \mu_0: \mathcal{B}(X) \rightarrow G \). This \( \mu \) is countably additive and regular \([8]\).

We shall prove some properties of \( \mu_0 \).

(a) First we prove that (II) holds when \( \{A_i\} \) are chosen from \( \mathcal{B}(X) \). Take any equicontinuous \( H \subset F' \). By (II) above there exists a \( p \in P \) and \( M, 0 < M < \infty \), such that \( \sup\{\sum f(\mu(A_i)(x_i)) : x_i \} \) a finite subset of \( E \) with \( p(x_i) \leq 1 \), and \( \{A_i\} \) a disjoint collection in \( \mathcal{U} \) \( \leq M \), for each \( f \in H \). Fix a finite subset \( \{x_i : 1 \leq i \leq n\} \) in \( E \) with \( p(x_i) \leq 1 \) for each \( i \), and a finite disjoint collection \( \{B_i\} \) in \( \mathcal{B}(X) \). Take nets in \( \mathcal{U} \), \( A^t_{\alpha} \rightarrow B_i \) in \( \mathcal{B}(X, \mathcal{T}) \). Put \( C^t_{\alpha} = A^t_{\alpha} \cup \bigcup_{i=1}^{t-1} A^t_{\alpha}, i \geq 2. \) Then \( \{C^t_{\alpha} : 1 \leq i \leq n\} \) are mutually disjoint and \( C^t_{\alpha} \rightarrow B_i \). From \( \sup\{\sum f(\mu(C^t_{\alpha})(x_i)) : f \in H, \{x_i\} \subset E \) with \( p(x_i) \leq 1 \} \leq M \), we get

\[ \sup \left\{ \left| \sum f(\mu_0(B_i)(x_i)) \right| : f \in H, \{x_i\} \subset E \right\} \leq M \]

for \( p(x_i) \leq 1 \) and \( \{B_i\} \) a disjoint finite collection in \( \mathcal{B}(X) \).
Now we claim that $\mu_0(B) \in \mathcal{L}(E, F)$ for every $B \in \mathcal{B}(X)$. Take a net $\{A_\alpha\}$ in $\mathcal{U}$ such that $A_\alpha \rightarrow B$ in $(\mathcal{B}(X), \mathcal{T})$. Fix $x \in E$. Since $\{\mu(A_\alpha)(x)\}_\alpha$ is a relatively $\mathcal{T}$-compact set we get $\mu_0(A)(x) \in F$. Using what is proved first in (a) we get $\mu_0(B) \in \mathcal{L}(E, F)$.

(b) Proceeding as in (a) we prove that (III) holds when $\{A_i\}$ are chosen from $\mathcal{B}(X)$.

Because of properties (a) and (u), the mapping $\mu_0: \mathcal{B}(X) \rightarrow \mathcal{L}(E, F)$ gives a $\mathcal{T}$-compact linear continuous mapping $L_1: (S(\mathcal{B}(X)) \otimes E, u) \rightarrow F$. The completion of $(S(\mathcal{B}(X)) \otimes E, u)$ contains $C(X) \otimes E$ and therefore also contains $C(X, E)$ (note $C(X) \otimes E$ is dense in $(C(X, E), u)$). Thus we get a continuous $\mathcal{T}$-compact operator $L_0: C(X, E) \rightarrow F$ (note $F$ is complete). For an $f \in C(Y)$, $x \in E$, and $g \in F'$, $g \circ L(f \otimes x) = g \circ \mu_1(f \otimes x)$. Take a sequence $\{f_n\}$ in $S(\mathcal{B}(Y))$, such that $f_n \rightarrow f$ uniformly on $Y$. Thus $\{f_n \circ \varphi\} \subset S(\mathcal{U})$ and $f_n \circ \varphi \rightarrow f \circ \varphi$ uniformly on $X$. This means

$$g \circ L_0(f \circ \varphi \otimes x) = \lim g \circ L_1(f_n \circ \varphi \otimes x) = \lim g \circ \mu_1(f_n \otimes x) = g \circ \mu_1(f \otimes x) = g \circ L(f \otimes x).$$

Thus $L_0(f \circ \varphi) = L(f)$. Since $C(Y) \otimes E$ is dense in $(C(Y, E), u)$ we get $L_0(f \circ \varphi) = L(f)$ for every $f \in C(Y, E)$. This proves the theorem.

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REFERENCES