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EXTENSIONS OF CERTAIN COMPACT OPERATORS ON VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. For any compact Hausdorff spaces X, Y with $\varphi\colon X \to Y$ a continuous onto mapping, E, F, Hausdorff locally convex spaces with F complete, $C(X,E)$ ($C(Y,E)$) all E-valued continuous functions on X (Y), and $L\colon C(Y,E) \to F$ a $\mathcal{T}$-compact continuous operator ($\sigma(F,F') \leq \mathcal{T} \leq \tau(F,F')$), it is proved there exists a $\mathcal{T}$-compact continuous operator $L_0\colon C(X,E) \to F$ such that $L_0(f \circ \varphi) = L(f)$ for every $f \in C(Y,E)$.

In this paper X, Y are compact Hausdorff spaces, $\varphi\colon X \to Y$ a continuous onto function, E, F Hausdorff locally convex spaces over K, the field of real or complex numbers, and $C(X,E)$ (resp. $C(Y,E)$) all continuous E-valued functions on X (resp. Y). The space $C(Y,E) \circ \varphi$ is a subspace of $C(X,E)$. When E, F are Banach spaces it is proved in [**1**, **2**] that every weakly compact operator $T\colon C(Y,E) \to F$ has extension to a weakly compact operator $T_0\colon C(X,E) \to F$ in the sense that $T_0(f \circ \varphi) = T(f)$ for every $f \in C(Y,E)$. Here we will prove the result for general locally convex spaces E, F assuming F to be complete, by using the measure extension techniques discussed in [**8**]. On $C(X,E)$ or $C(Y,E)$, u will denote the uniform topology. For locally convex spaces G_1, G_2, an operator $T\colon G_1 \to G_2$ will be called compact if bounded sets of G_1 are mapped into relatively compact subsets of G_2. For locally convex spaces we refer to [**9**]. $\mathcal{L}(E,F)$ will denote the space of all continuous linear operators from E to F. Let $\{|\cdot|_p\colon p \in P\}$ be the family of all continuous seminars on E. $M(X), M(Y)$ will denote all regular scalar Borel measures on X and Y resp. If $T\colon (C(Y,E),u) \to F$ is continuous and $f \in F'$, then $f \circ T \in (C(Y,E),u)'$ and so [**5**, **7**] there exists $p \in P$ such that $|f \circ T|_p \in M^+(Y)$ (note for $g \in C(Y)$, $g \geq 0$, $|(f \circ T)|_p(g) = \sup\{|(f \circ T)(h)|\colon h \in C(Y,E)$ and $\|h\|_p \leq f\}$, where $\|h\|_p(y) = \|h(y)\|_p$ [**7**]). Here $C(Y)$ stands for all K-valued continuous functions on Y. Also $(C(Y,E),u)' = M(Y,E')$ [**5**, **7**]. $\mathcal{B}(X), \mathcal{B}(Y)$ will denote all Borel subsets of X and Y respectively. For an algebra $\mathfrak{A}$ of subsets of a set Z, $S(\mathfrak{A})$ will denote all K-valued $\mathfrak{A}$-simple functions on Z.

THEOREM. *Assume F is a complete locally convex space, and let $\mathcal{T}$ be another locally convex topology on F such that $\sigma(F,F') \leq \mathcal{T} \leq \tau(F,F')$. Let $L\colon (C(Y,E),u) \to F$ be a continuous $\mathcal{T}$-compact operator, i.e., bounded subsets of $C(Y,E)$ are mapped into relatively $\mathcal{T}$-compact subsets of F. Then there exists a continuous*

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$\mathcal{T}$-compact operator L_0: $(C(X,E),u) \to F$ such that $L_0(f \circ \varphi) = L(f)$ for each $f \in C(Y,E)$.

PROOF. $\mathcal{L}(E,F)$ is the space of all continuous linear operators from E to F. Let $\tilde{F}$ be the weak completion of F and G the space of all continuous linear operators from E into $(\tilde{F}, \sigma(\tilde{F}, F'))$. For any finite subset $H \subset F'$ and any bounded $B \subset E$, a seminorm m is generated on G:

$$m(Q) = \sup\{|f \circ Q(x)| \colon x \in B,\ f \in H\},$$

$Q \in G$. Under the locally convex topology generated by these seminorms, G is a complete locally convex space. The topology on $\mathcal{L}(E,F)$ is the one induced by G. Since L is weakly compact, we get [**3**, **5**] a regular Borel measure μ_1: $\mathcal{B}(Y) \to \mathcal{L}(E,F)$ with the properties:

(I) For any $f \in F'$, $f \circ \mu_1 = f \circ L$.

(II) For any equicontinuous set $H \subset F'$, there exists a $p \in P$ such that

$$\sup\left\{\left|\sum f(\mu_1(A_i)(x_i))\right|\right\} < \infty,$$

where the supremum is taken over $f \in H$, all finite Borel partitions $\{A_i\}$ of Y, and all $x_i \in E$ satisfying $|x_i|_p \le 1$.

(III) For any bounded set $B \subset E$, $\{\sum \mu_1(A_i)x_i\}$, where $\{A_i\}$ varies over all finite disjoint collections of Borel subsets of X and $x_i \in B$, is relatively $\mathcal{T}$-compact in F.

Let $\{|\cdot|_s \colon s \in S\}$ be the family of all continuous seminorms on G. In the notation of [**8**, p. 160], let $\mathcal{U} = \{\varphi^{-1}(A) \colon A \in \mathcal{B}(Y)\}$; defines μ: $\mathcal{U} \to \mathcal{L}(E,F)$, $\mu(\varphi^{-1}(A)) = \mu_1(A)$, $A \in \mathcal{B}(Y)$. Each $s \in S$ gives an exhaustive, order σ-continuous submeasure $\dot{\mu}_s \colon \mathcal{U} \to [0,\infty)$, $\dot{\mu}_s(B) = \sup\{|\mu(A)|_s \colon A \in \mathcal{U}, A \subset B\}$ for every $B \in \mathcal{U}$. As in the proof of [**8**, Theorem 1, pp. 160–162], these submeasures can be extended to exhaustive, order σ-continuous, regular submeasures $\bar{\mu}_s \colon \mathcal{B}(X) \to [0,\infty)$ with the properties

(i) for any $s(1), s(2) \in S$, $\dot{\mu}_{s(1)} \le \dot{\mu}_{s(2)}$ implies $\bar{\mu}_{s(1)} \le \bar{\mu}_{s(2)}$,

(ii) for $\varepsilon > 0$, $s \in S$ and $B \in \mathcal{B}(X)$, there exists $B_0 \in \mathcal{U}$, such that $\bar{\mu}_s(B \triangle B_0) < \varepsilon$ (here $B \triangle B_0 = (B \backslash B_0) \cup (B_0 \backslash B)$).

On $\mathcal{B}(X)$ we define F-N topology $\mathcal{F}$ generated by $\{\bar{\mu}_s \colon s \in S\}$ [**4**, p. 271]. $\mathcal{B}(X)$ becomes a topological ring in which $\mathcal{U}$ is dense. This means the uniformly continuous mapping $\mu \colon \mathcal{U} \to G$ can be uniquely extended to a uniformly continuous mapping $\mu_0 \colon \mathcal{B}(X) \to G$. This μ is countably additive and regular [**8**].

We shall prove some properties of μ_0.

(a) First we prove that (II) holds when $\{A_i\}$ are chosen from $\mathcal{B}(X)$. Take any equicontinuous $H \subset F'$. By (II) above there exists a $p \in P$ and M, $0 < M < \infty$, such that $\sup\{|\sum f(\mu(A_i)(x_i))| \colon \{x_i\}$ a finite subset of E with $p(x_i) \le 1$, and $\{A_i\}$ a disjoint collection in $\mathcal{U}\} \le M$, for each $f \in H$. Fix a finite subset $\{x_i \colon 1 \le i \le n\}$ in E with $p(x_i) \le 1$ for each i, and a finite disjoint collection $\{B_i\}$ in $\mathcal{B}(X)$. Take nets in $\mathcal{U}$, $A^i_\alpha \to B_i$ in $(\mathcal{B}(X), \mathcal{F})$. Put $C^1_\alpha = A^1_\alpha$, $C^i_\alpha = A^i_\alpha \backslash \bigcup_{j=1}^{i-1} A^j_\alpha$, $i \ge 2$. Then $\{C^i_\alpha \colon 1 \le i \le n\}$ are mutually disjoint and $C^i_\alpha \to B_i$. From $\sup\{|\sum f(\mu(C^i_\alpha)(x_i))| \colon f \in H, \{x_i\} \subset E$ with $p(x_i) \le 1\} \le M$, we get

$$\sup\left\{\sum |f(\mu_0(B_i)(x_i))| \colon f \in H, \{x_i\} \subset E\right\} \le M$$

for $p(x_i) \le 1$ and $\{B_i\}$ a disjoint finite collection in $\mathcal{B}(X)$.

Now we claim that $\mu_0(B) \in \mathcal{L}(E,F)$ for every $B \in \mathcal{B}(X)$. Take a net $\{A_\alpha\}$ in $\mathcal{U}$ such that $A_\alpha \to B$ in $(\mathcal{B}(X), \mathcal{F})$. Fix $x \in E$. Since $\{\mu(A_\alpha)(x)\}_\alpha$ is a relatively $\mathcal{T}$-compact set we get $\mu_0(A)(x) \in F$. Using what is proved first in (a) we get $\mu_0(B) \in \mathcal{L}(E,F)$.

(b) Proceeding as in (a) we prove that (III) holds when $\{A_i\}$ are chosen from $\mathcal{B}(X)$.

Because of properties (a) and (ú), the mapping $\mu_0 \colon \mathcal{B}(X) \to \mathcal{L}(E,F)$ gives a $\mathcal{T}$-compact linear continuous mapping $L_1 \colon (S(\mathcal{B}(X)) \otimes E, u) \to F$. The completion of $(S(\mathcal{B}(X)) \otimes E, u)$ contains $C(X) \otimes E$ and therefore also contains $C(X,E)$ (note $C(X) \otimes E$ is dense in $(C(X,E), u)$). Thus we get a continuous $\mathcal{T}$-compact operator $L_0 \colon C(X,E) \to F$ (note F is complete). For an $f \in C(Y)$, $x \in E$, and $g \in F'$, $g \circ L(f \otimes x) = g \circ \mu_1(f \otimes x)$. Take a sequence $\{f_n\}$ in $S(\mathcal{B}(Y))$, such that $f_n \to f$ uniformly on Y. Thus $\{f_n \circ \varphi\} \subset S(\mathcal{U})$ and $f_n \circ \varphi \to f \circ \varphi$ uniformly on X. This means

$$\begin{aligned} g \circ L_0(f \circ \varphi \otimes x) &= \lim g \circ L_1(f_n \circ \varphi \otimes x) = \lim g \circ \mu_1(f_n \otimes x) \\ &= g \circ \mu_1(f \otimes x) = g \circ L(f \otimes x). \end{aligned}$$

Thus $L_0(f \circ \varphi) = L(f)$. Since $C(Y) \otimes E$ is dense in $(C(Y,E), u)$ we get $L_0(f \circ \varphi) = L(f)$ for every $f \in C(Y,E)$. This proves the theorem.

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