PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

AN ESTIMATE FOR CERTAIN MEROMORPHIC UNIVALENT FUNCTIONS

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(Communicated by Irwin Kra)

ABSTRACT. In this paper the coefficient problem for the family of univalent functions

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

in $\{|z| > 1\}$ has been studied. The author obtained the sharp estimate $b_7 \leq 1/4 + 3/280$ when $g(z)$ is an odd function and all its coefficients are real.

Let Σ_0 denote the class of functions $g(z)$ univalent in $|z| > 1$, regular apart from a simple pole at the point at infinity and having the expansion at that point

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}.$$

Garabedian and Schiffer [1] obtained the sharp estimate $|b_3| \leq (1 + 2e^{-6})/2$, and at the same time they remarked that if all the coefficients b_n of $g(z)$ are real, then $b_3 \leq 1/2$. Furthermore, Kubota [2] obtained an estimate of b_5. He proved the following theorem.

THEOREM. *If all the coefficients b_n of $g(z)$ are real, then*

$$b_5 \leq \frac{1}{3} + \frac{4}{507}$$

with equality holding only for the function $\tilde{g}(z)$ which satisfies the algebraic equation

$$\left(w^2 + \frac{12}{13}\right)^3 = \left(z^3 + \frac{6}{13}z + \frac{6}{13}z^{-1} + z^{-3}\right)^2, \qquad w = \tilde{g}(z).$$

The expansion of $\tilde{g}(z)$ at the point at infinity begins

$$z - \frac{4}{13}z^{-1} + \frac{16}{169}z^{-3} + \left(\frac{1}{3} + \frac{4}{507}\right)z^{-5} + \cdots.$$

As $b_2 = b_4 = 0$ for $\tilde{g}(z)$, it is easy to verify by induction that $\tilde{g}(z)$ is an odd function in Σ_0.

In this paper we shall be concerned with the coefficient b_7 and prove the following.

Received by the editors April 28, 1986 and, in revised form, October 17, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30A32; Secondary 30A34.
Key words and phrases. Univalent, coefficient, trajectory, quadratic differential.

THEOREM 1. *If $F(z) = z + \sum_{n=1}^{\infty} b_{2n-1} z^{-2n+1}$ is an odd function in Σ_0 and all its coefficients b_{2n-1} are reals, then*

$$b_7 \leq \tfrac{1}{4} + \tfrac{3}{280}$$

with equality holding only for the functions $\tilde{F}_j(z)$ which satisfy the algebraic equations

$$w^4 + t_j w^2 = G_j(z), \qquad w = \tilde{F}_j(z), \quad j = 1, 2, \tag{1}$$

where

$$t_1 = -5\sqrt{\tfrac{6}{36}}, \qquad t_2 = 5\sqrt{\tfrac{6}{35}},$$

$$G_1(z) = z^4 - \sqrt{\tfrac{6}{35}} z^2 - \tfrac{26}{35} - \sqrt{\tfrac{6}{35}} z^{-2} + z^{-4},$$

$$G_2(z) = z^4 + \sqrt{\tfrac{6}{35}} z^2 - \tfrac{26}{35} + \sqrt{\tfrac{6}{35}} z^{-2} + z^{-4}.$$

The expansions of $\tilde{F}_j(z)$ at the point at infinity begin

$$\tilde{F}_1(z) = z + \sqrt{\tfrac{6}{35}} z^{-1} - \tfrac{1}{70} z^{-3} - \tfrac{1}{5}\sqrt{\tfrac{6}{35}} z^{-5} + (\tfrac{1}{4} + \tfrac{3}{280}) z^{-7} + \cdots,$$

$$\tilde{F}_2(z) = z - \sqrt{\tfrac{6}{35}} z^{-1} - \tfrac{1}{70} z^{-3} + \tfrac{1}{5}\sqrt{\tfrac{6}{35}} z^{-5} + (\tfrac{1}{4} + \tfrac{3}{280}) z^{-7} + \cdots.$$

Our proof is due to Goluzin's inequality.

Firstly we give two lemmas which will be used later.

Let σ_m denote the class of all analytic and m-valent functions for $|z| > 1$ with a development

$$F(z) = \sum_{n=-m}^{\infty} b_n z^{-n}, \qquad b_{-m} \neq 0. \tag{2}$$

LEMMA 1 [**3**]. *Suppose $F \in \sigma_m$, and $P_q(w) = \sum_{n=0}^{q} a_n w^n$ is an arbitrary nonconstant polynomial of degree q. Then $P_q(F(z)) \in \sigma_{mq}$.*

LEMMA 2 [**4**]. *If $F \in \sigma_m$ with the development* (2), *then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq \sum_{n=1}^{m} n|b_{-n}|^2. \tag{3}$$

The inequalities that arise from (3) are known as the Goluzin inequalities.

Now we shall prove Theorem 1. Set

$$\begin{aligned} F(z)^2 &= z^2 + \sum_{n=0}^{\infty} c_{2n} z^{-2n}, \\ F(z)^4 &= z^4 + \sum_{n=-1}^{\infty} d_{2n} z^{-2n}. \end{aligned} \tag{4}$$

Obviously, in our case all coefficients c_{2n} and d_{2n} are real. The relations between the coefficients are

$$\begin{aligned} c_0 &= 2b_1, \\ c_2 &= b_1^2 + 2b_3, \\ c_4 &= 2b_1 b_3 + 2b_5, \\ c_6 &= b_3^2 + 2b_1 b_5 + 2b_7, \end{aligned} \tag{5}$$

and

$$
(6)\qquad \begin{aligned} d_{-2} &= 2c_0, \\ d_0 &= c_0^2 + 2c_2, \\ d_2 &= 2c_0c_2 + 2c_4, \\ d_4 &= c_2^2 + 2c_0c_4 + 2c_6. \end{aligned}
$$

We introduce a real parameter t and apply Lemma 2 to the function $F(z)^4 + tF(z)^2$, which is 4-valent with a pole of order four at infinity. This gives

$$
(7)\qquad (c_2t + d_2)^2 + 2(c_4t + d_4)^2 \le (t + d_{-2})^2 + 2.
$$

After rearrangement, (7) becomes

$$
(1 - c_2^2 - 2c_4^2)t^2 + 2(d_{-2} - c_2d_2 - 2c_4d_4)t + 2 + d_{-2}^2 - d_2^2 - 2d_4^2 \ge 0.
$$

Applying Lemma 2 to the function $F(z)^2$ which belongs to σ_2, we find that the coefficient of t^2 in (7) is nonnegative. The positive definiteness of the Hermitian form implies

$$
(8)\qquad (d_{-2} - c_2d_2 - 2c_4d_4)^2 \le (1 - c_2^2 - 2c_4^2)(2 + d_{-2}^2 - d_2^2 - 2d_4^2).
$$

We shall show that d_4 lies inside a certain circle:

$$
(9)\qquad |d_4 - \omega| \le R.
$$

To do so we bring (8) into the form

$$
\left[d_4 - \frac{c_4(d_{-2} - c_2d_2)}{1 - c_2^2}\right]^2 \le \frac{(1 - c_2^2 - 2c_4^2)(2 + d_{-2}^2 - d_2^2)}{2(1 - c_2^2)} - \frac{(d_{-2} - c_2d_2)^2}{2(1 - c_2^2)} + \frac{c_4^2(d_{-2} - c_2d_2)^2}{(1 - c_2^2)^2}.
$$

With the aid of (6) we thus have

$$
(10)\qquad \omega = 2c_0c_4 - \frac{2c_2c_4^2}{1 - c_2^2},
$$

and upon using the identity

$$
(d_{-2} - c_2d_2)^2 - (c_2d_{-2} - d_2)^2 = (d_{-2}^2 - d_2^2)(1 - c_2^2),
$$

it turns out that R has the value

$$
(11)\qquad R = \frac{1 - c_2^2 - 2c_4^2}{1 - c_2^2}.
$$

From (6), (10), and (11), inequality (9) implies

$$
(12)\qquad c_6 \le \frac{1}{2} - \frac{1}{2}c_2^2 - \frac{c_4^2}{1 - c_2}.
$$

Substituting the relations (5), we find that (12) becomes

$$
(13)\qquad b_7 \le \tfrac{1}{4} + \Phi(b_1, b_3, b_5),
$$

where

$$
\Phi(b_1, b_3, b_5) = -\frac{1}{2}b_3^2 - \frac{1}{4}(b_1^2 + 2b_3)^2 - b_1b_5 - \frac{2(b_1b_3 + b_5)^2}{1 - b_1^2 - 2b_3}.
$$

Simple computing shows that the function $\Phi(b_1, b_3, b_5)$ achieves its maximum at the points

$$P_1\left(\sqrt{\tfrac{6}{35}}, -\tfrac{1}{70}, -\tfrac{1}{5}\sqrt{\tfrac{6}{35}}\right) \quad \text{and} \quad P_2\left(-\sqrt{\tfrac{6}{35}}, -\tfrac{1}{70}, \tfrac{1}{5}\sqrt{\tfrac{6}{35}}\right).$$

Hence we have

$$\begin{aligned} b_7 &\le \tfrac{1}{4} + \Phi\left(\sqrt{\tfrac{6}{35}}, -\tfrac{1}{70}, -\tfrac{1}{5}\sqrt{\tfrac{6}{35}}\right) \\ &= \tfrac{1}{4} + \Phi\left(-\sqrt{\tfrac{6}{35}}, -\tfrac{1}{70}, \tfrac{1}{5}\sqrt{\tfrac{6}{35}}\right) = \tfrac{1}{4} + \tfrac{3}{280}. \end{aligned}$$

Next, the equality occurs only for

$$(b_1, b_3, b_5) = P_j, \qquad j = 1, 2, \tag{14}$$

and

$$d_4 = \omega + R. \tag{15}$$

In this case the equality of (7) holds for some $t = t_j$ and

$$c_{2n} t_j + d_{2n} = 0, \qquad n \ge 3, \ j = 1, 2.$$

Substitute (5), (6), (14), and (15) in (7). We find

$$\begin{aligned} t_1 &= -5\sqrt{\tfrac{6}{35}}, & t_2 &= 5\sqrt{\tfrac{6}{35}}, \\ t_1 + d_{-2} &= -\sqrt{\tfrac{6}{35}}, & t_2 + d_{-2} &= \sqrt{\tfrac{6}{35}}, \\ c_0 t_1 + d_0 &= -\tfrac{26}{35}, & c_0 t_2 + d_0 &= -\tfrac{26}{35}, \\ c_2 t_1 + d_2 &= -\sqrt{\tfrac{6}{35}}, & c_2 t_2 + d_2 &= \sqrt{\tfrac{6}{35}}, \\ c_4 t_1 + d_4 &= 1, & c_4 t_2 + d_4 &= 1. \end{aligned}$$

Thus (1) is true.

Finally, we shall prove that the $\tilde{F}_j(z)$ $(j = 1, 2)$ belong to Σ_0.

Let $Q^*(w; t_j)\, dw^2$ $(j = 1, 2)$ be the quadratic differential

$$w^2 (w^2 + \tfrac{1}{2} t_j)^2\, dw^2, \qquad j = 1, 2.$$

For each value of j there are four end domains E_{1j}^*, E_{2j}^*, E_{3j}^*, E_{4j}^*, in the trajectory structure of $Q^*(w; t_j)\, dw^2$ on the upper half w-plane. For a suitable determination each of the functions

$$\varsigma = \int w(w^2 + \tfrac{1}{2} t_j)\, dw, \qquad j = 1, 2, \tag{16}$$

maps E_{1j}^*, E_{2j}^*, E_{3j}^*, E_{4j}^* respectively onto an upper half-plane, a lower half-plane, an upper half-plane and a lower half-plane, the positive real axis, corresponding to the half-infinite segment $\operatorname{Im} \varsigma = 0$, $\frac{1}{16} t_j^2 < \operatorname{Re} \varsigma < \infty$.

On the other hand, for each value of j there are four end domains E_{1j}, E_{2j}, E_{3j}, E_{4j} in the trajectory structure of the quadratic differential

$$\begin{aligned} & z^{-10}(z-1)^2(z+1)^2(z-i)^2(z+i)^2(z-e^{i\theta_j})^2 \\ & \cdot (z+e^{i\theta_j})^2(z-e^{-i\theta_j})^2(z+e^{-i\theta_j})^2\, dz^2, \qquad j = 1, 2, \end{aligned}$$

$$\cos 2\theta_1 = \tfrac{1}{4}\sqrt{\tfrac{6}{35}}, \qquad \cos 2\theta_2 = -\tfrac{1}{4}\sqrt{\tfrac{6}{35}},$$

on the domain $|z| > 1$, $\operatorname{Im} z > 0$. For a suitable determination each of the functions

$$\varsigma = \int z^{-5}(z-1)(z+1)(z-i)(z+i)(z-e^{i\theta_j}) \cdot (z+e^{i\theta_j})(z-e^{-i\theta_j})(z+e^{-i\theta_j})\,dz, \qquad j=1,2, \tag{17}$$

maps E_{1j}, E_{2j}, E_{3j}, E_{4j} respectively onto an upper half-plane, a lower half-plane, an upper half-plane and a lower half-plane, the points 1, $e^{i\theta_j}$, i, $-e^{-i\theta_j}$, -1 corresponding to the points

$$\tfrac{1}{2} - 2\cos 2\theta_j + \tfrac{23}{280}, \qquad -\tfrac{1}{2} - \cos^2 2\theta_j + \tfrac{23}{280},$$
$$\tfrac{1}{2} + 2\cos 2\theta_j + \tfrac{23}{280}, \qquad -\tfrac{1}{2} - \cos^2 2\theta_j + \tfrac{23}{280}, \qquad \tfrac{1}{2} - 2\cos 2\theta_j + \tfrac{23}{280}.$$

Thus we can combine functions (16) with (17) respectively for $j = 1,2$ to obtain two functions, each of which maps the domain $|z| > 1$, $\operatorname{Im} z > 0$ into the upper half w-plane. By reflection these functions extend to the functions $\tilde{F}_j(z)$ ($j = 1,2$) respectively, which map $|z| > 1$ onto a domain admissible with respect to $Q^*(w;t_j)\,dw^2$ ($j = 1,2$). It means that $\tilde{F}_j(z)$ ($j = 1,2$) belong to Σ_0. This completes the proof of Theorem 1.

REFERENCES

1. P. R. Garabedian and M. Schiffer, *A coefficient inequality for schlicht functions*, Ann. of Math. (2) **61** (1955), 116–136.
2. Y. Kubota, *A coefficient inequality for certain meromorphic univalent functions*, Kodai Math. Sem. Rep. **26** (1974), 85–94.
3. Н. А. Лебедев, ***Принцип площадей в теории однолистных функции***, "Nauka", Moscow, 1975, pp. 86–88.
4. L. V. Ahlfors, *Conformal invariants*, McGraw-Hill, 1972, pp. 86–87.

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