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WEIGHTED NORM INEQUALITIES FOR MULTIPLIERS

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ABSTRACT. We consider the two-weight function problem for a class of multiplier operators that include the Riesz and Bessel potentials.

1. Let $\varphi(t)$ be a nonnegative function on $(0,\infty)$ such that

$$m(\xi)=\int_0^\infty \varphi(t^{1/2})e^{-t\xi}\frac{dt}{t}$$

is finite for every $\xi>0$. We define the operator T by

$$(Tf)^\wedge(x)=m(\pi|x|^2)\hat{f}(x), \tag{1.1}$$

where $\hat{f}(x)=\int_{\mathbf{R}^n} f(y)e^{-2\pi i x\cdot y}\,dy$. In this note we prove weighted norm inequalities for the class of multiplier operators defined by (1.1). This class includes the Riesz potentials and the Bessel potentials by taking $\varphi(t)=t^\alpha$ and $\varphi(t)=t^\alpha e^{-t^2}$, $\alpha>0$, respectively. The case of the Riesz potentials has been studied in [**5**] and [**3**].

A weight function u is said to belong to D_μ, $\mu\geq 1$, if $u(B_{ts}(x))\leq ct^{n\mu}u(B_s(x))$ for every $t>1$, $s>0$ and $x\in\mathbf{R}^n$, where $B_s(x)$ denotes the ball with center x and radius s and $u(B_s(x))$ its u-measure. We write $D_\infty=\bigcup_{\mu\geq 1}D_\mu$. Analogously, $u\in RD_\nu$, $\nu>0$, if $u(B_{st}(x))\geq ct^{n\nu}u(B_s(x))$ for every $t\geq 1$, $s>0$ and $x\in\mathbf{R}^n$. It is not hard to see that if $u\in D_\infty$ then $u\in RD_\nu$ for some $\nu>0$. L^p_u is the class of functions g such that $\|g\|_{p,u}=(\int_{\mathbf{R}^n}|g(x)|^p u(x)\,dx)^{1/p}$ is finite. $S_{0,0}$ will denote the class of Schwartz functions whose Fourier transforms have compact support not including the origin. The space H^p_u, $0<p<\infty$, consists of all tempered distributions f such that $\|f\|_{H^p_u}=\|N(f)\|_{p,u}$ is finite, where $N(f)(x)=\sup_{|x-y|\leq t}|F(y,t)|$, $F(x,t)=f*\Psi_t(x)$, $\Psi_t(\cdot)=t^{-n}\Psi(\cdot/t)$, $\Psi\in S$. If $u\in D_\infty$, $\|f\|_{H^p_u}$ is equivalent to $\|N_0(f)\|_{p,u}$, where $N_0(f)(x)=\sup_{t>0}|F(x,t)|$ (see [**8**]).

We prove the following

THEOREM 1. *Let $0<p<q<\infty$, $u\in D_\mu$, $v\in D_\infty\cap RD_\nu$ with $\nu q>\mu p$. If*

$$\varphi(t)u(B_t(x))^{1/q}\leq cv(B_t(x))^{1/p} \tag{1.2}$$

for every $t>0$ and every $x\in\mathbf{R}^n$ then

$$\|Tf\|_{q,u}\leq c\|f\|_{H^p_v}\quad \textit{for every } f\in S_{0,0}.$$

THEOREM 2. *Let $0<q\leq 1$ and $u,v\in D_\infty$. If*

$$u(Q)\int_0^l\left(\int_s^{2s}\varphi(t)\frac{dt}{t}\right)^q\frac{ds}{s}\leq cv(Q)$$

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for every cube Q (l is the edgelength of Q), then

$$\|Tf\|_{q,u} \le c\|f\|_{H^q_v} \quad \text{for every } f \in S_{0,0}.$$

If $1 < p < \infty$ and v is such that $\|\cdot\|_{p,v} \approx \|\cdot\|_{H^p_v}$, Theorem 1 gives results for L^p. (For such v see [**1**].) The method used to prove our theorems is similar to that in [**5**] and [**3**]. The unweighted case has been considered in [**6**]. When $v \equiv 1$ and $p = q$ a characterization of u in terms of a maximal function associated with the kernel of T is given in [**7**].

2. We state two lemmas which will be needed later.

LEMMA (2.1). *Let σ be a measure on $\mathbf{R}^{n+1}_+$ and $v \in D_\infty$ on $\mathbf{R}^n$. Then for $0 < p \le q < \infty$,*

$$\left(\int_{\mathbf{R}^{n+1}_+} |F(x,t)|^q \, d\sigma(x,t)\right)^{1/q} \le c\left(\int_{\mathbf{R}^n} N(f)(x)^p v(x)\, dx\right)^{1/p}$$

for every measurable f on $\mathbf{R}^n$, if and only if $\sigma(B_Q) \le cv(Q)^{q/p}$ for every cube Q where $B_Q = \{(x,t)\colon x \in Q,\ 0 \le t \le l\}$, l is the edgelength of Q.

See [**4**] for a proof.

LEMMA (2.2). *Let $0 < q < \infty$ and $G(x,t)$ be a solution of the heat equation in $\mathbf{R}^{n+1}_+$. Then*

$$|G(x,t)|^q \le \frac{C_q}{t^{(n+2)/2}} \int_{\substack{|x-y|<t^{1/2}\\ t/2<s<t}} |G(y,s)|^q \, dy\, ds.$$

The last follows by applying Lemma 2 of [**2**] with $r^2 = t/(1+n)$.

3. Let $\varphi(t)$ be a nonnegative function on $t > 0$, $\Psi(x) = e^{-\pi|x|^2}$ and $F(x,t) = f * \Psi_t(x)$. For $u \in D_\infty$ and $\varepsilon \in \mathbf{R}$ we define

$$M_{\varepsilon,u}(f)(x) = \sup_{t>0} \varphi(t) u(B_t(x))^{\varepsilon/n} |F(x,t)|.$$

Theorem 1 will be a consequence of the following lemmas.

LEMMA (3.1). *Let $0 < p \le q < \infty$, $\varepsilon > 0$, and $u, v \in D_\infty$ satisfying (1.2). Then for every $f \in S_{0,0}$ we have*

$$|Tf(x)| \le c\|f\|^\theta_{H^p_v} M_{-\varepsilon,u}(f)(x)^{1-\theta},$$

where $\theta = \varepsilon/(\varepsilon + n/q)$.

PROOF. For $f \in S_{0,0}$ we have that

$$Tf(x) = 2\int_0^\infty \varphi(t) F(x,t)\, \frac{dt}{t}. \tag{3.2}$$

In fact, if distance(supp $\hat{f}, 0) = \lambda > 0$ then we have $|F(x,t)| \le e^{-\pi(t\lambda)^2}\|\hat{f}\|_1$ and so (3.2) follows by taking Fourier transforms.

Let $\delta > 0$; we write

$$Tf(x) = 2\left(\int_0^\delta + \int_\delta^\infty\right)\varphi(t)F(x,t)\frac{dt}{t} = I_1 + I_2.$$

Then

$$\begin{aligned}|I_1| &\le 2\int_0^\delta \varphi(t)u(B_t(x))^{-\varepsilon/n}u(B_t(x))^{\varepsilon/n}|F(x,t)|\frac{dt}{t}\\ &\le 2M_{-\varepsilon,u}(f)(x)\int_0^\delta u(B_t(x))^{\varepsilon/n}\frac{dt}{t}.\end{aligned}$$

Since $|F(x,t)| \le N(f)(y)$ for $|x-y| < t$ we get that $|F(x,t)| \le v(B_t(x))^{-1/p}\|f\|_{H_v^p}$ and then by (1.2) we have

$$|I_2| \le c\|f\|_{H_v^p}\int_\delta^\infty u(B_t(x))^{-1/q}\frac{dt}{t}.$$

Since $u \in D_\infty$ implies $u \in RD_\beta$ for some $\beta > 0$, the lemma then follows by estimating the integrals and minimizing in δ as in Lemma (3.1) of [**5**].

LEMMA (3.3). *Let $0 < p < q < \infty$, $\varepsilon > 0$, $u \in D_\mu$ and $v \in D_\infty \cap RD_\nu$ with $\nu q > \mu p$. If*

$$\varphi(t)u(B_t(x))^{1/q-\varepsilon/n} \le cv(B_t(x))^{1/p}, \qquad t > 0,\ x \in \mathbf{R}^n, \tag{3.4}$$

then

$$\|M_{-\varepsilon,u}f\|_{q,u} \le c\|f\|_{H_v^p}.$$

PROOF. The function G defined by $G(x,t) = F(x,(4\pi t)^{1/2})$ is a solution of the heat equation $\partial/\partial t - \Delta = 0$ on $\mathbf{R}_+^{n+1}$. Hence by (3.4) and Lemma (2.2) we have

$$\begin{aligned}&|\varphi(t)u(B_t(x))^{-\varepsilon/n}F(x,t)|^q\\ &\qquad\le cv(B_t(x))^{q/p}u(B_t(x))^{-1}\frac{1}{t^{n+2}}\int_{\substack{|x-y|<t\\ t/\sqrt{2}<s<t}}|F(y,s)|^q s\,dy\,ds.\end{aligned}$$

Since $v \in D_\infty$ we have

$$M_{-\varepsilon,u}(f)(x)^q \le c\int_{|x-y|<\sqrt{2}s}|F(y,s)|^q v(B_s(x))^{q/p}u(B_s(x))^{-1}s^{-n-1}\,dy\,ds.$$

Integrating this inequality with respect to $u(x)\,dx$ we get

$$\|M_{-\varepsilon,u}f\|_{q,u}^q \le c\int_{\mathbf{R}_+^{n+1}}|F(y,s)|^q\,d\sigma(y,s),$$

where $d\sigma(y,s) = \{s^{-n-1}\int_{|x-y|<\sqrt{2}s}v(B_s(x))^{q/p}u(B_s(x))^{-1}u(x)\,dx\}\,dy\,ds$. We will show that σ is a Carleson measure with respect to v, i.e.

$$\int_0^l\int_Q d\sigma(y,s) \le cv(Q)^{q/p} \tag{3.5}$$

for every cube Q with edgelength l and therefore the lemma follows by Lemma (2.1). If we set

$$I = \int_0^l v(B_s(x))^{q/p} u(B_s(x))^{-1} \frac{ds}{s}$$

then $I \leq cv(B_l(x))^{q/p} u(B_l(x))^{-1}$. In fact, since $u \in D_\mu$ and $v \in RD_\nu$ we have $u(B_s(x)) \geq c(s/l)^{n\mu} u(B_l(x))$ and $v(B_s(x)) \leq c(s/l)^{n\nu} v(B_l(x))$, $0 < s < l$. Hence

$$I < cl^{-n\nu q/p + n\mu} \left(\int_0^l s^{-1+n\nu q/p - n\mu} \, ds \right) v(B_l(x))^{q/p} u(B_l(x))^{-1}.$$

Since

$$\int_0^l \int_Q d\sigma(y,s) \leq \int_{3Q} u(x) I \, dx$$

and $\nu q/p - \mu > 0$, (3.5) then follows.

PROOF OF THEOREM 1. By Lemma (3.1) we have

$$\|Tf\|_{q,u}^q \leq c \|f\|_{H_v^p}^{\theta q} \|M_{-\varepsilon,u}(f)\|_{(1-\theta)q,u}^{(1-\theta)q},$$

with $\theta = \varepsilon/(\varepsilon + n/q)$. For $\varepsilon > 0$ small enough we have $(1-\theta)q > p$ and $\nu(1-\theta)q > \mu p$ and since $1/(1-\theta)q - \varepsilon/n = 1/q$ we may apply Lemma (3.3) and then $\|M_{-\varepsilon,u} f\|_{(1-\theta)q,u} \leq C \|f\|_{H_v^p}$.

4. Proof of Theorem 2. We shall first show that

$$|Tf(x)|^q \leq c \int_{|x-y|<\sqrt{2}s} |F(y,s)|^q s^{-n-1} \left(\int_s^{2s} \varphi(t) \frac{dt}{t} \right)^q dy \, ds. \tag{4.1}$$

As in the proof of the Lemma (3.3), by Lemma (2.2) we have

$$|F(x,t)| \leq c \left(\int_{\substack{|x-y|<t \\ t/\sqrt{2}<s<t}} |F(y,s)|^q s^{-n-1} \, dy \, ds \right)^{1/q}.$$

If $\chi_{B(t,x)}$ is the characteristic function of $B(t,x) = \{(y,s) \colon |x-y| < s,\ t/\sqrt{2} < s < t\}$ then since $0 < q \leq 1$ by (3.2) and Minkowsky's integral inequality we get

$$|Tf(x)|^q \leq c \int_{|x-y| \leq \sqrt{2}s} |F(y,s)|^q s^{-n-1} \left(\int_0^\infty \varphi(t) \chi_{B(t,x)}(y,s) \frac{dt}{t} \right)^q dy \, ds.$$

Since for $|x-y| < \sqrt{2}s$,

$$\int_0^\infty \varphi(t) \chi_{B(t,x)}(y,s) \frac{dt}{t} \leq \int_s^{2s} \varphi(t) \frac{dt}{t}$$

then (4.1) follows. Now integrating (4.1) with respect to $u(x)\,dx$ we obtain

$$\|Tf\|_{q,u}^q \leq c \int_{\mathbf{R}_+^{n+1}} |F(y,s)|^q \sigma(y,s) \, dy \, ds$$

where

$$\sigma(y,s) = s^{-n-1} \left(\int_s^{2s} \varphi(t) \frac{dt}{t} \right)^q \left(\int_{|x-y|<\sqrt{2}s} u(x) \, dx \right).$$

By the hypothesis made on u and v we have that σ is a Carleson measure with respect to v and so by Lemma (1.2) the theorem follows.

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