WEIGHTED NORM INEQUALITIES FOR MULTIPLIERS
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ABSTRACT. We consider the two-weight function problem for a class of multiplier operators that include the Riesz and Bessel potentials.

1. Let \( \varphi(t) \) be a nonnegative function on \((0, \infty)\) such that
\[
m(t) = \int_0^\infty \varphi(t^{1/2}) e^{-t} \frac{dt}{t}
\]
is finite for every \( \xi > 0 \). We define the operator \( T \) by
\[
(Tf)(x) = m(\pi |x|^2) f(x),
\]
where \( f(x) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i x \cdot y} dy \). In this note we prove weighted norm inequalities for the class of multiplier operators defined by (1.1). This class includes the Riesz potentials and the Bessel potentials by taking \( \varphi(t) = ta \) and \( \varphi(t) = tae^{-t^2} \), \( a > 0 \), respectively. The case of the Riesz potentials has been studied in [5] and [3].

A weight function \( u \) is said to belong to \( D_\mu, \mu \geq 1 \), if \( u(B_{ts}(x)) < ct^{\mu u}u(B_{s}(x)) \) for every \( t > 1, s > 0 \) and \( x \in \mathbb{R}^n \), where \( B_{s}(x) \) denotes the ball with center \( x \) and radius \( s \) and \( u(B_{s}(x)) \) its \( u \)-measure. We write \( D_\infty = \bigcup_{\mu \geq 1} D_\mu \). Analogously, \( u \in RD_\nu, \nu > 0 \), if \( u(B_{st}(x)) > ct^{\mu u}u(B_{s}(x)) \) for every \( t \geq 1, s > 0 \) and \( x \in \mathbb{R}^n \).

It is not hard to see that if \( u \in D_\infty \) then \( u \in RD_\nu \) for some \( \nu > 0 \). \( L^p_u \) is the class of functions \( g \) such that \( \|g\|_{p,u} = \left( \int_{\mathbb{R}^n} |g(x)|^p u(x) dx \right)^{1/p} \) is finite. \( S_{0,0} \) will denote the class of Schwartz functions whose Fourier transforms have compact support not including the origin. The space \( H^p_u, 0 < p < \infty \), consists of all tempered distributions \( f \) such that \( \|f\|_{H^p_u} = ||N(f)||_{p,u} \) is finite, where \( N(f)(x) = \sup_{|x-y| \leq t} |F(y,t)|, F(x,t) = f \ast \psi(t), \psi(t) = t^{-n} e^{-t/2} \), \( \psi \in \mathcal{S} \). If \( u \in D_\infty \), \( \|f\|_{H^p_u} \) is equivalent to \( \|N_0(f)\|_{p,u} \), where \( N_0(f)(x) = \sup_{t>0} |F(x,t)| \) (see [8]).

We prove the following

**Theorem 1.** Let \( 0 < p < q < \infty \), \( u \in D_\mu, v \in D_\infty \cap RD_\nu \) with \( \nu q > \mu p \). If
\[
\varphi(t)u(B_t(x))^{1/q} \leq ct^{\nu q}u(B_t(x))^{1/p}
\]
for every \( t > 0 \) and every \( x \in \mathbb{R}^n \) then
\[
\|Tf\|_{q,u} \leq c\|f\|_{H^p_u} \quad \text{for every } f \in S_{0,0}.
\]

**Theorem 2.** Let \( 0 < q \leq 1 \) and \( u, v \in D_\infty \). If
\[
u(Q) \int_0^1 \left( \int_s^{2s} \varphi(t) \frac{dt}{t} \right)^q \frac{ds}{s} \leq c v(Q)
\]

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for every cube $Q$ ($l$ is the edgelength of $Q$), then

$$\|Tf\|_{p,u} \leq c\|f\|_{H^p}$$

for every $f \in S_{0,0}$.

If $1 < p < \infty$ and $v$ is such that $\| \cdot \|_{p,v} \approx \| \cdot \|_{H^p}$, Theorem 1 gives results for $L^p$. (For such $v$ see [1].) The method used to prove our theorems is similar to that in [5] and [3]. The unweighted case has been considered in [6]. When $v \equiv 1$ and $p = q$ a characterization of $u$ in terms of a maximal function associated with the kernel of $T$ is given in [7].

2. We state two lemmas which will be needed later.

**LEMMA (2.1).** Let $\sigma$ be a measure on $\mathbb{R}^{n+1}_+$ and $v \in D^p$ on $\mathbb{R}^n$. Then for $0 < p < q < \infty$,

$$\left( \int_{\mathbb{R}^{n+1}_+} |F(x,t)|^q \, d\sigma(x,t) \right)^{1/q} \leq c \left( \int_{\mathbb{R}^n} N(f)(x)^p v(x) \, dx \right)^{1/p}$$

for every measurable $f$ on $\mathbb{R}^n$, if and only if $c(\sigma) < cv(Q)^{q/p}$ for every cube $Q$ where $B_Q = \{(x,t): x \in Q, \ 0 < t < l\}$, $l$ is the edgelength of $Q$.


**LEMMA (2.2).** Let $0 < q < \infty$ and $G(x,t)$ be a solution of the heat equation in $\mathbb{R}^{n+1}_+$. Then

$$|G(x,t)|^q \leq \frac{C_q}{(n+2)\sqrt{t}} \int_{|x-y| < t^{1/2}} |G(y,s)|^q \, dy \, ds.$$  

The last follows by applying Lemma 2 of [2] with $r^2 = t/(1 + n)$.

3. Let $\varphi(t)$ be a nonnegative function on $t > 0$, $\Psi(x) = e^{-\pi|z|^2}$ and $F(x,t) = f * \Psi_t(x)$. For $u \in D_\infty$ and $\varepsilon \in \mathbb{R}$ we define

$$M_{\varepsilon,u}(f)(x) = \sup_{t > 0} \varphi(t)u(B_t(x))^{\varepsilon/n} |F(x,t)|.$$  

Theorem 1 will be a consequence of the following lemmas.

**LEMMA (3.1).** Let $0 < p \leq q < \infty$, $\varepsilon > 0$, and $u, v \in D_\infty$ satisfying (1.2). Then for every $f \in S_{0,0}$ we have

$$|Tf(x)| \leq c\|f\|_{H^p} M_{-\varepsilon,u}(f)(x)^{1-\theta},$$

where $\theta = \varepsilon/(\varepsilon + n/q)$.

**PROOF.** For $f \in S_{0,0}$ we have that

$$Tf(x) = 2 \int_0^\infty \varphi(t)F(x,t) \, dt.$$  

In fact, if distance(supp $\hat{f},0) = \lambda > 0$ then we have $|F(x,t)| \leq e^{-\pi(t\lambda)^2} \|\hat{f}\|_1$ and so (3.2) follows by taking Fourier transforms.
Let \( \delta > 0 \); we write
\[
T_f(x) = 2 \left( \int_0^\delta + \int_\delta^\infty \right) \varphi(t) F(x, t) \frac{dt}{t} = I_1 + I_2.
\]

Then
\[
|I_1| \leq 2 \int_0^\delta \varphi(t) u(B_t(x))^{-\varepsilon/n} u(B_t(x))^{\varepsilon/n} |F(x, t)| \frac{dt}{t} \\
\leq 2M_{-\varepsilon, u}(f)(x) \int_0^\delta u(B_t(x))^{\varepsilon/n} \frac{dt}{t}.
\]

Since \( |F(x, t)| \leq N(f)(y) \) for \( |x-y| < t \) we get that \( |F(x, t)| \leq v(B_t(x))^{-1/p} \|f\|_{H_v^p} \)
and then by (1.2) we have
\[
|I_2| \leq c\|f\|_{H_v^p} \int_0^\infty u(B_t(x))^{-1/q} \frac{dt}{t}.
\]

Since \( u \in D_\infty \) implies \( u \in RD_\beta \) for some \( \beta > 0 \), the lemma then follows by estimating the integrals and minimizing in \( \delta \) as in Lemma (3.1) of [5].

**Lemma (3.3).** Let \( 0 < p < q < \infty, \varepsilon > 0, u \in D_\mu \) and \( v \in D_\infty \cap RD_\nu \) with \( \nu q > \mu p \). If
\[
\varphi(t)u(B_t(x))^{1/q-\varepsilon/n} \leq cv(B_t(x))^{1/p}, \quad t > 0, \ x \in \mathbb{R}^n,
\]
then
\[
\|M_{-\varepsilon, u}f\|_{q,u} \leq c\|f\|_{H_v^p}.
\]

**Proof.** The function \( G \) defined by \( G(x, t) = F(x, (4\pi t)^{1/2}) \) is a solution of the heat equation \( \partial/\partial t - \Delta = 0 \) on \( \mathbb{R}^{n+1}_+ \). Hence by (3.4) and Lemma (2.2) we have
\[
|\varphi(t)u(B_t(x))^{-\varepsilon/n} F(x, t)|^q \\
\leq cv(B_t(x))^{q/p}u(B_t(x))^{-1/q} \int_{|x-y|<t} |F(y,s)|^q s \ dy ds.
\]

Since \( v \in D_\infty \) we have
\[
M_{-\varepsilon, u}(f)(x)^q \leq c \int_{|x-y|<\sqrt{2s}} |F(y,s)|^q u(B_s(x)) \ dy ds.
\]

Integrating this inequality with respect to \( u(x) \, dx \) we get
\[
\|M_{-\varepsilon, u}f\|_{q,u} \leq c \int_{\mathbb{R}^{n+1}} |F(y,s)|^q d\sigma(y,s),
\]
where \( d\sigma(y,s) = \{s^{-n-1} \int_{|x-y|<\sqrt{2s}} v(B_s(x))^{q/p}u(B_s(x))^{-1}u(x) \, dx \} dy ds \). We will show that \( \sigma \) is a Carleson measure with respect to \( v \), i.e.
\[
\int_0^t \int_Q d\sigma(y,s) \leq cv(Q)^{q/p}
\]
for every cube $Q$ with edgelength $l$ and therefore the lemma follows by Lemma (2.1). If we set
\[
I = \int_0^l v(B_l(x))^{q/p} u(B_l(x))^{-1} \frac{ds}{s}
\]
then $I \leq cv(B_l(x))^{q/p} u(B_l(x))^{-1}$. In fact, since $u \in D_\mu$ and $v \in RD_\nu$ we have $u(B_l(x)) \geq c(s/l)^{\mu}u(B_l(x))$ and $v(B_l(x)) \leq c(s/l)^{\nu}v(B_l(x))$, $0 < s < l$. Hence
\[
I < cl^{-\nu q/p + \mu} \left( \int_0^l s^{-1 + \nu q/p - \mu} ds \right) v(B_l(x))^{q/p} u(B_l(x))^{-1}.
\]
Since
\[
\int_1^l \int_Q d\sigma(y, s) \leq \int_3 Q u(x) I \, dx
\]
and $\nu q/p - \mu > 0$, (3.5) then follows.

**Proof of Theorem 1.** By Lemma (3.1) we have
\[
\|Tf\|_{q, u}^\theta \leq c \|f\|_{H_\theta}^\theta \|M_{-\theta, u}(f)\|_{(1-\theta)^2}^\theta,
\]
with $\theta = \epsilon/(\epsilon + n/q)$. For $\epsilon > 0$ small enough we have $(1 - \theta)q > p$ and $\nu(1 - \theta)q > \mu p$ and since $1/(1 - \theta)q - \epsilon/n = 1/q$ we may apply Lemma (3.3) and then $\|M_{-\theta, u}(f)\|_{(1-\theta)^2}^\theta \leq C \|f\|_{H_\theta}$.

**4. Proof of Theorem 2.** We shall first show that
\[
\|Tf\|_{q, u}^\theta \leq c \left( \int_{|x-y|<\sqrt{2}s} |F(y,s)|^q s^{-n-1} \left( \int_s^{2s} \varphi(t) \frac{dt}{t} \right)^q dy \, ds. \right.
\]
As in the proof of the Lemma (3.3), by Lemma (2.2) we have
\[
|F(x,t)| \leq c \left( \int_{|x-y|<t} |F(y,s)|^q s^{-n-1} dy \, ds \right)^{1/q}.
\]
If $\chi_{B(t,x)}$ is the characteristic function of $B(t,x) = \{(y,s) : |x-y| < s, t/\sqrt{2} < s < t \}$ then since $0 < q \leq 1$ by (3.2) and Minkowsky’s integral inequality we get
\[
|Tf(x)|^q \leq c \int_{|x-y|<\sqrt{2}s} |F(y,s)|^q s^{-n-1} \left( \int_0^\infty \varphi(t) \chi_{B(t,x)}(y,s) \frac{dt}{t} \right)^q dy \, ds.
\]
Since for $|x-y| < \sqrt{2}s$,
\[
\int_0^\infty \varphi(t) \chi_{B(t,x)}(y,s) \frac{dt}{t} \leq \int_s^{2s} \varphi(t) \frac{dt}{t} - \int_0^t \varphi(t) \frac{dt}{t}
\]
then (4.1) follows. Now integrating (4.1) with respect to $u(x) \, dx$ we obtain
\[
\|Tf\|_{q, u}^\theta \leq c \int_{R_n^{\theta+1}} |F(y,s)|^q \sigma(y,s) dy \, ds
\]
where
\[
\sigma(y,s) = s^{-n-1} \left( \int_s^{2s} \varphi(t) \frac{dt}{t} \right)^q \left( \int_{|x-y|<\sqrt{2}s} u(x) \, dx \right).
\]
By the hypothesis made on $u$ and $v$ we have that $\sigma$ is a Carleson measure with respect to $v$ and so by Lemma (1.2) the theorem follows.
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