A CHARACTERIZATION OF $L^p$-IMPROVING MEASURES

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ABSTRACT. A Borel measure $\mu$ on a compact abelian group $G$ is $L^p$-improving if $\mu$ convolves $L^p(G)$ to $L^{p+\varepsilon}(G)$ for some $\varepsilon > 0$. We characterize $L^p$-improving measures by means of their Fourier transforms.

Introduction. Let $G$ be a compact abelian group, $\Gamma$ its discrete dual group, and $m$ normalized Haar measure on $G$. A Borel measure $\mu$ is said to be $L^p$-improving for some $p$, $1 < p < \infty$, if there are constants $\varepsilon > 0$ and $K$ so that whenever $f \in L^p(G)$, $\|\mu * f\|_{p+\varepsilon} \leq K\|f\|_p$. Since $\mu * L^1 \subseteq L^1$ and $\mu * L^\infty \subseteq L^\infty$, an application of the complex interpolation theorem shows that if $\mu$ is $L^p$-improving for some $p$, then $\mu$ is $L^p$-improving for all $1 < p < \infty$.

Stein in [10, pp. 122-123] posed the problem of characterizing $L^p$-improving measures by the "size" of the measure $\mu$. We provide such a characterization in terms of the size of the sets

$E(\varepsilon) \equiv \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geq \varepsilon\}$

for $\varepsilon > 0$.

To make clear our notion of "size," we recall the following definition of $\Lambda(p)$ set, which was introduced by Rudin in [9] for subsets of $\mathbb{Z}$.

For $E \subseteq \Gamma$, $\text{Trig}_E(G)$ will denote the set of $E$-polynomials, i.e., the set of integrable functions $f: G \to \mathbb{C}$ with $\text{supp}(f)$ a finite subset of $E$. Let $2 < p < \infty$. A subset $E$ of $\Gamma$ is called a $\Lambda(p)$ set if there is a constant $c$ so that whenever $f \in \text{Trig}_E(G)$, $\|f\|_p \leq c\|f\|_2$. The least such constant $c$ is called the $\Lambda(p)$ constant for $E$ and is denoted by $\Lambda(p, E)$. For standard results on $\Lambda(p)$ sets we refer the reader to [9, 4].

We will show that a measure $\mu$ is $L^p$-improving if and only if the sets $E(\varepsilon)$, with $\varepsilon > 0$, are $\Lambda(p)$ for all $2 < p < \infty$, with certain $\Lambda(p)$ constants.

An example of an $L^p$-improving measure on the circle is the Riesz product $\mu = \prod_{k=1}^\infty (1 + (e^{3^k t} + e^{-3^k t}/2))$ [2]. It is easy to see that $\{n: |\hat{\mu}(n)| = 1/2^m\}$ is precisely

$E_m = \left\{ \sum_{i=1}^m \varepsilon_i 3^{j_i}: \varepsilon_i = \pm 1, j_i \in \mathbb{Z}^+ \text{ and } j_i \neq j_k \text{ if } i \neq k \right\}$.

Bonami [2] proved that such sets were $\Lambda(p)$ for all $p > 2$, with $\Lambda(p, E_m) \leq A^m p^{m/2}$. Here $A$ does not depend on $p$ or $m$. This example was the motivation for our characterization of $L^p$-improving measures.

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Other examples of $L^p$-improving measures include any $L^q(G)$ function for $q > 1$ (this follows from Young's inequality), any measure $\mu$ on the circle group satisfying $|\hat{\mu}(n)| = O(n^{-\alpha})$ for $\alpha > 0$ [11, p. 127], and Cantor-Lebesgue measures associated with Cantor sets having constant ratio of dissection [3] (see also [1, 5]). Building on the work of [2], Ritter [6] characterized all $L^p$-improving Riesz products by means of their Fourier transforms, and in particular showed that all Riesz products on the circle are $L^p$-improving. We will use some of the methods of [2 and 6] in proving our theorem.

**Main result.**

Let $\mu$ be a Borel measure on $G$ with $\|\mu\| \leq 1$. The following are equivalent.

1. $\mu$ is $L^p$-improving.
2. There are constants $p > 2$ and $\alpha > 1$ so that for every $\varepsilon > 0$, $E(\varepsilon)$ is a $\Lambda(p)$ set with $\Lambda(p, E(\varepsilon)) = O(\varepsilon^{-\alpha})$.
3. Each of the sets $E(\varepsilon)$, $\varepsilon > 0$, is a $\Lambda(q)$ set for all $2 < q < \infty$, and there is a constant $c$ such that $\Lambda(q, E(\varepsilon)) = O(q^{-c \log \varepsilon / \varepsilon})$.

**Proof.** (1) $\Rightarrow$ (2) Since $\mu$ is $L^p$-improving, we may assume there are constants $p > 2$ and $K$ so that $\|\mu * f\|_p \leq K\|f\|_2$ whenever $f \in L^2(G)$.

Let $\varepsilon > 0$. For $f$ an $E(\varepsilon)$-polynomial, define $g$ by
\[
g(\gamma) = \begin{cases} \hat{f}(\gamma) / \hat{\mu}(\gamma) & \text{for } \gamma \in E(\varepsilon), \\ 0 & \text{otherwise.} \end{cases}
\]

Then $\mu * g = f$ and $\|g\|_2 \leq \|f\|_2 / \varepsilon$. Hence
\[
\|f\|_p = \|\mu * g\|_p \leq K\|g\|_2 \leq K\|f\|_2 / \varepsilon.
\]

Thus $E(\varepsilon)$ is a $\Lambda(p)$ set with $\Lambda(p, E(\varepsilon)) \leq K / \varepsilon$.

(2) $\Rightarrow$ (1) For $j \geq 1$ let $E_j = \{\gamma : 1 / 2^j < |\hat{\mu}(\gamma)| \leq 1 / 2^{j-1}\}$. Certainly $\text{supp} \hat{\mu} \subseteq \bigcup_{j=1}^\infty E_j$, and by (2) there is a constant $K$ so that each set $E_j$ is a $\Lambda(p)$ set with $\Lambda(p, E_j) \leq 2^{j\alpha} K$. A standard duality argument shows that if $1/p + 1/p' = 1$ and $f \in L^2(G)$ then
\[
\sum_{\gamma \in E_j} |\hat{f}(\gamma)|^2 \leq (2^{j\alpha} K)^2 \|f\|_{p'}^2.
\]

Let $\mu^N$ denote the $N$th convolution power of $\mu$. Clearly $|\hat{\mu}^N(\gamma)| \leq 2^{-(j-1)N}$ on $E_j$. Thus for $f \in L^2(G)$ we have
\[
\|\mu^N * f\|_2^2 = \sum_{j=1}^\infty \sum_{\gamma \in E_j} |\hat{\mu}^N(\gamma)|^2 |\hat{f}(\gamma)|^2 \leq \sum_{j=1}^\infty \frac{1}{2^{2N(j-1)}} (2^{j\alpha} K)^2 \|f\|_{p'}^2 \leq C\|f\|_{p'}^2,
\]

provided $N$ is sufficiently large. It follows that $\mu^N$ is $L^p$-improving for sufficiently large $N$. As Ritter [6] has proven that $\mu$ is $L^p$-improving if and only if $\mu^N$ is $L^p$-improving, this concludes the proof of (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (2) is clear.

Before proving (1) $\Rightarrow$ (3), we prove a lemma of independent interest.
Lemma. Suppose \( \|\mu\| \leq 1 \), and for some \( p > 2 \) and constant \( K \), \( \|\mu * f\|_p \leq K\|f\|_2 \) for all \( f \in L^2(G) \). Let \( p(n) = p^{n+1}/2^n \) and \( s(n) = \sum_{j=0}^{n}(2/p)^j \). Then whenever \( f \in L^2(G) \),

\[
(*)_{n} \quad \|\mu^{n+1} * f\|_{p(n)} \leq K^{s(n)}\|f\|_2.
\]

Proof. We proceed inductively. Certainly \( (*)_0 \) holds, so assume \( (*)_{n-1} \) is satisfied. Let \( t(n) = (2/p)^n \). Since the norm of \( \mu \) as a convolution map from \( L^2 \) to \( L^p \) is at most \( K \), and the norm of \( \mu \) from \( L^\infty \) to \( L^\infty \) is at most \( \|\mu\| \leq 1 \), the complex interpolation method shows that for each integer \( n \geq 0 \),

\[
\|\mu * f\|_{p(n)} \leq K^{s(n)}\|f\|_{p(n-1)}
\]

for \( f \in L^p(n-1)(G) \).

By the induction assumption, \( \mu^n * f \in L^p(n-1)(G) \) whenever \( f \in L^2(G) \); thus

\[
\|\mu^{n+1} * f\|_{p(n)} \leq K^{s(n)}\|\mu^n * f\|_{p(n-1)} \leq K^{s(n)}\|f\|_2.
\]

Proof of Theorem (ctd.). \( (1) \Rightarrow (3) \) We will continue to use the functions \( p(n) \) and \( s(n) \) as defined in the previous lemma.

Given \( q, 2 < q < \infty \), choose an integer \( n \geq 0 \) so that \( p(n-1) < q < p(n) \).

Observe that \( E(\varepsilon) = \{\gamma: |\hat{\mu}^{n+1}(\gamma)| \geq \varepsilon^{n+1}\} \); thus the proof of \( (1) \Rightarrow (2) \), together with the lemma, shows that for \( \varepsilon > 0 \), \( \Lambda(p(n), E(\varepsilon)) \leq K^{s(n)}/\varepsilon^{n+1} \). Without loss of generality, we may assume \( K \geq 1 \).

It follows that if \( f \) is an \( E(\varepsilon) \)-polynomial and \( s = \sum_{j=0}^{\infty}(2/p)^j = 1/(1 - (2/p)) \), then

\[
\|f\|_q \leq \|f\|_{p(n)} \leq K^{s(n)}\|f\|_2 \leq K^s\frac{1}{\varepsilon^{n+1}}\|f\|_2.
\]

Let \( c = 1/\log(p/2) \). Since \( n < \log q/\log(p/2) \), the inequality above shows that

\[
\|f\|_q \leq K^s\frac{1}{\varepsilon q^c} \|f\|_2
\]

whenever \( f \in \text{Trig}_{E(\varepsilon)}(G) \). This establishes \( (3) \).

Applications.

Corollary 1. If \( \mu \) is a Borel measure on \( G \) and \( \sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^r < \infty \) for some \( r < \infty \), then \( \mu \) is \( L^p \)-improving.

Remark. This includes the case of \( |\hat{\mu}(\gamma)| = O(n^{-\alpha}) \), \( \alpha > 0 \).

Proof. Since \( \sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^r \geq \sum_{\gamma \in E(\varepsilon)} |\hat{\mu}(\gamma)|^r \geq \varepsilon^r |E(\varepsilon)| \), \( E(\varepsilon) \) is a finite set for all \( \varepsilon > 0 \), hence a \( \Lambda(p) \) set for all \( p > 2 \) with \( \Lambda(p, E(\varepsilon)) \leq O(\varepsilon^{-r}) \).

It is known [1] that if \( \mu \) is a probability measure on the circle which is \( L^p \)-improving, then \( \sup_{n \neq 0} |\hat{\mu}(\gamma)| < 1 \). This is not true for other groups. However we do have

Corollary 2. Let \( 2 < p < \infty \). If \( \mu \) convolves \( L^2(G) \) to \( L^p(G) \), then

\[
\limsup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)| \leq \sqrt{2/p}\|\mu\|.
\]

Proof. It is shown in [9, 3.4] that if an infinite set \( E \subseteq \Gamma \) is a \( \Lambda(p) \) set then

\[
\Lambda(p, E) \geq O(\sqrt{p}).
\]

If \( \varepsilon > \sqrt{2/p}\|\mu\| \), this fact, together with \( (3) \) of the main theorem, shows that \( E(\varepsilon) \) must be a finite set.
REMARK. By duality $\mu$ convolves $L^2$ to $L^p$ for some $p > 2$ if and only if $\mu$ convolves $L^{p'}$ to $L^2$, where $1/p + 1/p' = 1$. Thus Corollary 2 may be restated as

**COROLLARY 2'.** Let $1 < p < 2$. If $\mu$ convolves $L^p$ to $L^2$, then

$$\limsup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)| \leq \sqrt{2 - (2/p)}\|\mu\|.$$  

**COROLLARY 3.** If $\mu$ convolves $L^2$ to $\bigcap_{1<p<\infty} L^p$, or equivalently, $\mu$ maps $\bigcup_{1<p<2} L^p$ to $L^2$, then $\limsup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)| = 0$.

Corollary 3 answers a question posed by McGehee, which was communicated to us by Graham.

**COROLLARY 4.** If a measure $\mu$ has the property that $\inf\{|\hat{\mu}(\gamma)|: \hat{\mu}(\gamma) \neq 0\} > 0$, then $\mu$ is $L^p$-improving if and only if the cardinality of the support of $\hat{\mu}$ is finite.

**PROOF.** Sufficiency is clear. For necessity note that the hypotheses imply that $\text{supp} \hat{\mu}$ is contained in a $\Lambda(p)$ set for some $p > 2$. A basic property of $\Lambda(p)$ sets is that such measures are actually $L^p$ functions [4]; so $\limsup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)| = 0$.

Much is known about the structure of $\Lambda(p)$ sets (cf., [4, Chapter 6 and 9]). To cite but one example: it is known that if $A$ is an arithmetic progression of length $N$, and $E$ is a $\Lambda(p)$ set for some $p > 2$, then $|A \cap E| \leq CA(p,E)^2N^{2/p}$, where $C$ is a constant independent of $N$, $E$ and $p$ [9, 3.5]. (Here $|\cdot|$ denotes the cardinality of the set.)

Thus if $\mu$ is an $L^p$-improving measure and $\|\mu\| \leq 1$, then for each $0 < \epsilon \leq 1$ and $2 < p < \infty$

$$|A \cap E(\epsilon)| \leq C_1 \frac{C_2}{\epsilon^2} p^{-2C_2 \log \epsilon N^{2/p}},$$  

where $C_1$ and $C_2$ are constants independent of $p, \epsilon$ and $N$. Taking

$$p = (-1/C_2 \log \epsilon) \log N$$

we obtain

**COROLLARY 5.** Let $\mu$ be an $L^p$-improving measure with $\|\mu\| \leq 1$. There are constants $C_1$ and $C_2$, independent of $N$, so that if $A$ is an arithmetic progression of length $N$, then

$$|A \cap E(\epsilon)| \leq C_1 (\log N)^{-2C_2 \log \epsilon}.$$  

A measure $\mu$, acting as a convolution operator from $L^1$ to $L^1$, is said to be an Enflo operator if there is a subspace $Y$ of $L^1$, isomorphic to $L^1$, on which $\mu$ is an isomorphism. In [8] Rosenthal proves that if for each $\epsilon > 0$, $\{\gamma: |\hat{\mu}(\gamma)| > \epsilon\}$ is a $\Lambda(p)$ set for some $p > 2$, then the measure $\mu$ is non-Enflo. Consequently, all $L^p$-improving measures are non-Enflo.

We will say that a measure $\mu$ has property $(\ast)$ if whenever $R$ is an infinite dimensional reflexive subspace of $L^1$, and $\mu|_R$ is an isomorphism onto its range, then $R$ is isomorphic to a Hilbert space. Rosenthal asks in [8] if there are any measures $\mu$ which have property $(\ast)$ and for which $\limsup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)| \neq 0$. Our final proposition answers this question affirmatively.
PROPOSITION. Any $L^p$-improving measure has property (*).

REMARKS 1. This is a generalization of the fact that the $E(\varepsilon)$ sets for an $L^p$-improving measure are $\Lambda(p)$ for all $p < \infty$, so that $L^{p}_{E(\varepsilon)} \cong L^2$.

2. Rosenthal has communicated to us that Bourgain, in an unpublished proof, showed that Riesz products have property (*).

PROOF. Let $\mu$ be an $L^p$-improving measure and $R$ an infinite dimensional reflexive subspace of $L^1$. Then $R$ is isomorphic to a subspace of $L^p$ for some $p > 1$ [7]. Fix $1 < r < p$ and choose $R_1 \cong R$ as in [8] so that $R_1 \subseteq L^r$ and $\mu|_{R_1}$ is an isomorphism onto its range. $R$ is closed in $L^1$; hence $R_1$ is closed in $L^r$ and $\mu \ast R_1$ is a closed subspace of $L^1$.

Since $\mu$ is $L^p$-improving there is a constant $\delta > 0$ so that $\mu \ast L^r \subseteq L^{r+\delta}$. In particular, $\mu \ast R_1 \subseteq L^{r+\delta}$. Since $\mu \ast R_1$ is closed in $L^1$, it is closed in $L^{r+\delta}$, and thus $R$ is isomorphic to a closed subspace of $L^{r+\delta}$.

Fix $r_1$, with $r < r_1 < r + \delta$ and let $s = (r + \delta)/r$. Let $r(n) = r_{1}^{n}/r_{n+1}^{n-1}$. The complex interpolation method shows that $\mu \ast L^{r(n)} \subseteq L^{r(n)}s$ for all $n \geq 0$. Suppose we inductively assume that $R \cong R_{n+1}$, a closed subspace of $L^{r(n)}$, and $\mu|_{R_{n+1}}$ is an isomorphism onto its range. Then $R \cong \mu \ast R_{n+1}$, a closed subspace of $L^{r(n)s}$, and since $r(n+1) < r(n)s$, there is a closed subspace $R_{n+2}$ of $L^{r(n+1)}$ isomorphic to $R$ on which $\mu$ is an isomorphism. If $n$ is chosen so that $r(n) \geq 2$, then $R$ is isomorphic to $\mu \ast R_{n+1}$, a closed subspace of $L^2$, proving the result.

In conclusion, we would like to thank C. Graham for introducing us to the notion of $L^p$-improving measures.

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