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A CHARACTERIZATION OF L^p-IMPROVING MEASURES

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ABSTRACT. A Borel measure μ on a compact abelian group G is L^p-improving if μ convolves $L^p(G)$ to $L^{p+\varepsilon}(G)$ for some $\varepsilon > 0$. We characterize L^p-improving measures by means of their Fourier transforms.

Introduction. Let G be a compact abelian group, Γ its discrete dual group, and m normalized Haar measure on G. A Borel measure μ is said to be L^p*-improving* for some p, $1 < p < \infty$, if there are constants $\varepsilon > 0$ and K so that whenever $f \in L^p(G)$, $\|\mu * f\|_{p+\varepsilon} \leq K\|f\|_p$. Since $\mu * L^1 \subseteq L^1$ and $\mu * L^\infty \subseteq L^\infty$, an application of the complex interpolation theorem shows that if μ is L^p-improving for some p, then μ is L^p-improving for all $1 < p < \infty$.

Stein in [**10**, pp. 122–123] posed the problem of characterizing L^p-improving measures by the "size" of the measure μ. We provide such a characterization in terms of the size of the sets

$$E(\varepsilon) \equiv \{\gamma \in \Gamma : |\hat{\mu}(\gamma)| \geq \varepsilon\}$$

for $\varepsilon > 0$.

To make clear our notion of "size," we recall the following definition of $\Lambda(p)$ set, which was introduced by Rudin in [**9**] for subsets of $\mathbf{Z}$.

For $E \subseteq \Gamma$, $\mathrm{Trig}_E(G)$ will denote the set of E-polynomials, i.e., the set of integrable functions $f : G \to \mathbf{C}$ with $\mathrm{supp}(\hat{f})$ a finite subset of E. Let $2 < p < \infty$. A subset E of Γ is called a $\Lambda(p)$ *set* if there is a constant c so that whenever $f \in \mathrm{Trig}_E(G)$, $\|f\|_p \leq c\|f\|_2$. The least such constant c is called the $\Lambda(p)$ *constant for* E and is denoted by $\Lambda(p, E)$. For standard results on $\Lambda(p)$ sets we refer the reader to [**9**, **4**].

We will show that a measure μ is L^p-improving if and only if the sets $E(\varepsilon)$, with $\varepsilon > 0$, are $\Lambda(p)$ for all $2 < p < \infty$, with certain $\Lambda(p)$ constants.

An example of an L^p-improving measure on the circle is the Riesz product $\mu = \prod_{k=1}^{\infty}(1+(e^{i3^k t}+e^{-i3^k t}/2))$ [**2**]. It is easy to see that $\{n : |\hat{\mu}(n)| = 1/2^m\}$ is precisely

$$E_m = \left\{ \sum_{i=1}^{m} \varepsilon_i 3^{j_i} : \varepsilon_i = \pm 1,\ j_i \in \mathbf{Z}^+ \text{ and } j_i \neq j_k \text{ if } i \neq k \right\}.$$

Bonami [**2**] proved that such sets were $\Lambda(p)$ for all $p > 2$, with $\Lambda(p, E_m) \leq A^m p^{m/2}$. Here A does not depend on p or m. This example was the motivation for our characterization of L^p-improving measures.

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Other examples of L^p-improving measures include any $L^q(G)$ function for $q > 1$ (this follows from Young's inequality), any measure μ on the circle group satisfying $|\hat{\mu}(n)| = O(n^{-\alpha})$ for $\alpha > 0$ [**11**, p. 127], and Cantor-Lebesgue measures associated with Cantor sets having constant ratio of dissection [**3**] (see also [**1**, **5**]). Building on the work of [**2**], Ritter [**6**] characterized all L^p-improving Riesz products by means of their Fourier transforms, and in particular showed that all Riesz products on the circle are L^p-improving. We will use some of the methods of [**2** and **6**] in proving our theorem.

Main result.

THEOREM. *Let μ be a Borel measure on G with $\|\mu\| \leq 1$. The following are equivalent.*

(1) *μ is L^p-improving.*

(2) *There are constants $p > 2$ and $\alpha \geq 1$ so that for every $\varepsilon > 0$, $E(\varepsilon)$ is a $\Lambda(p)$ set with $\Lambda(p, E(\varepsilon)) = O(\varepsilon^{-\alpha})$.*

(3) *Each of the sets $E(\varepsilon)$, $\varepsilon > 0$, is a $\Lambda(q)$ set for all $2 < q < \infty$, and there is a constant c such that $\Lambda(q, E(\varepsilon)) = O(q^{-c \log \varepsilon}/\varepsilon)$.*

PROOF. (1)$\Rightarrow$(2) Since μ is L^p-improving, we may assume there are constants $p > 2$ and K so that $\|\mu * f\|_p \leq K\|f\|_2$ whenever $f \in L^2(G)$.

Let $\varepsilon > 0$. For f an $E(\varepsilon)$-polynomial, define g by

$$\hat{g}(\gamma) = \begin{cases} \hat{f}(\gamma)/\hat{\mu}(\gamma) & \text{for } \gamma \in E(\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu * g = f$ and $\|g\|_2 \leq \|f\|_2/\varepsilon$. Hence

$$\|f\|_p = \|\mu * g\|_p \leq K\|g\|_2 \leq \frac{K}{\varepsilon}\|f\|_2.$$

Thus $E(\varepsilon)$ is a $\Lambda(p)$ set with $\Lambda(p, E(\varepsilon)) \leq K/\varepsilon$.

(2)$\Rightarrow$(1) For $j \geq 1$ let $E_j = \{\gamma \leq \Gamma : 1/2^j < |\hat{\mu}(\gamma)| \leq 1/2^{j-1}\}$. Certainly $\operatorname{supp} \hat{\mu} \subseteq \bigcup_{j=1}^{\infty} E_j$, and by (2) there is a constant K so that each set E_j is a $\Lambda(p)$ set with $\Lambda(p, E_j) \leq 2^{j\alpha}K$. A standard duality argument shows that if $1/p + 1/p' = 1$ and $f \in L^2(G)$ then

$$\sum_{\gamma \in E_j} |\hat{f}(\gamma)|^2 \leq (2^{j\alpha}K)^2 \|f\|_{p'}^2.$$

Let μ^N denote the Nth convolution power of μ. Clearly $|\hat{\mu}^N(\gamma)| \leq 2^{-(j-1)N}$ on E_j. Thus for $f \in L^2(G)$ we have

$$\begin{aligned} \|\mu^N * f\|_2^2 &= \sum_{j=1}^{\infty} \sum_{\gamma \in E_j} |\hat{\mu}^N(\gamma)|^2 |\hat{f}(\gamma)|^2 \\ &\leq \sum_{j=1}^{\infty} \frac{1}{2^{2N(j-1)}} (2^{j\alpha}K)^2 \|f\|_{p'}^2 \leq C\|f\|_{p'}^2, \end{aligned}$$

provided N is sufficiently large. It follows that μ^N is L^p- improving for sufficiently large N. As Ritter [**6**] has proven that μ is L^p-improving if and only if μ^N is L^p-improving, this concludes the proof of (2)$\Rightarrow$(1).

(3)$\Rightarrow$(2) is clear.

Before proving (1)$\Rightarrow$(3), we prove a lemma of independent interest.

LEMMA. *Suppose $\|\mu\| \leq 1$, and for some $p > 2$ and constant K, $\|\mu * f\|_p \leq K\|f\|_2$ for all $f \in L^2(G)$. Let $p(n) = p^{n+1}/2^n$ and $s(n) = \sum_{j=0}^{n}(2/p)^j$. Then whenever $f \in L^2(G)$,*

$$(*_n) \qquad \|\mu^{n+1} * f\|_{p(n)} \leq K^{s(n)}\|f\|_2.$$

PROOF. We proceed inductively. Certainly $(*_0)$ holds, so assume $(*_{n-1})$ is satisfied. Let $t(n) = (2/p)^n$. Since the norm of μ as a convolution map from L^2 to L^p is at most K, and the norm of μ from L^∞ to L^∞ is at most $\|\mu\| \leq 1$, the complex interpolation method shows that for each integer $n \geq 0$,

$$\|\mu * f\|_{p(n)} \leq K^{t(n)}\|f\|_{p(n-1)}$$

for $f \in L^{p(n-1)}(G)$.

By the induction assumption, $\mu^n * f \in L^{p(n-1)}(G)$ whenever $f \in L^2(G)$; thus

$$\|\mu^{n+1} * f\|_{p(n)} \leq K^{t(n)}\|\mu^n * f\|_{p(n-1)} \leq K^{s(n)}\|f\|_2.$$

PROOF OF THEOREM (CTD.). (1)⇒(3) We will continue to use the functions $p(n)$ and $s(n)$ as defined in the previous lemma.

Given q, $2 < q < \infty$, choose an integer $n \geq 0$ so that $p(n-1) < q \leq p(n)$.

Observe that $E(\varepsilon) = \{\gamma : |\hat{\mu}^{n+1}(\gamma)| \geq \varepsilon^{n+1}\}$; thus the proof of (1)⇒(2), together with the lemma, shows that for $\varepsilon > 0$, $\Lambda(p(n), E(\varepsilon)) \leq K^{s(n)}/\varepsilon^{n+1}$. Without loss of generality, we may assume $K \geq 1$.

It follows that if f is an $E(\varepsilon)$-polynomial and $s = \sum_{j=0}^{\infty}(2/p)^j = 1/(1-(2/p))$, then

$$\|f\|_q \leq \|f\|_{p(n)} \leq \frac{K^{s(n)}}{\varepsilon^{n+1}}\|f\|_2 \leq \frac{K^s}{\varepsilon^{n+1}}\|f\|_2.$$

Let $c = 1/\log(p/2)$. Since $n < \log q/\log(p/2)$, the inequality above shows that

$$\|f\|_q \leq \frac{K^s}{\varepsilon} q^{-c\log\varepsilon}\|f\|_2$$

whenever $f \in \mathrm{Trig}_{E(\varepsilon)}(G)$. This establishes (3).

Applications.

COROLLARY 1. *If μ is a Borel measure on G and $\sum_{\gamma\in\Gamma}|\hat{\mu}(\gamma)|^r < \infty$ for some $r < \infty$, then μ is L^p-improving.*

REMARK. This includes the case of $|\hat{\mu}(n)| = O(n^{-\alpha})$, $\alpha > 0$.

PROOF. Since $\sum_{\gamma\in\Gamma}|\hat{\mu}(\gamma)|^r \geq \sum_{\gamma\in E(\varepsilon)}|\hat{\mu}(\gamma)|^r \geq \varepsilon^r|E(\varepsilon)|$, $E(\varepsilon)$ is a finite set for all $\varepsilon > 0$, hence a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p, E(\varepsilon)) \leq O(\varepsilon^{-r})$.

It is known [**1**] that if μ is a probability measure on the circle which is L^p-improving, then $\sup_{n\neq 0}|\hat{\mu}(n)| < 1$. This is not true for other groups. However we do have

COROLLARY 2. *Let $2 < p < \infty$. If μ convolves $L^2(G)$ to $L^p(G)$, then*

$$\limsup_{\gamma\in\Gamma}|\hat{\mu}(\gamma)| \leq \sqrt{2/p}\|\mu\|.$$

PROOF. It is shown in [**9**, 3.4] that if an infinite set $E \subseteq \Gamma$ is a $\Lambda(p)$ set then $\Lambda(p, E) \geq O(\sqrt{p})$. If $\varepsilon > \sqrt{2/p}\|\mu\|$, this fact, together with (3) of the main theorem, shows that $E(\varepsilon)$ must be a finite set.

REMARK. By duality μ convolves L^2 to L^p for some $p > 2$ if and only if μ convolves $L^{p'}$ to L^2, where $1/p + 1/p' = 1$. Thus Corollary 2 may be restated as

COROLLARY 2'. *Let* $1 < p < 2$. *If* μ *convolves* L^p *to* L^2, *then*

$$\limsup_{\gamma\in\Gamma} |\hat{\mu}(\gamma)| \le \sqrt{2-(2/p)}\|\mu\|.$$

COROLLARY 3. *If* μ *convolves* L^2 *to* $\bigcap_{2<p<\infty} L^p$, *or equivalently,* μ *maps* $\bigcup_{1<p<2} L^p$ *to* L^2, *then* $\limsup_{\gamma\in\Gamma} |\hat{\mu}(\gamma)| = 0$.

Corollary 3 answers a question posed by McGehee, which was communicated to us by Graham.

COROLLARY 4. *If a measure* μ *has the property that* $\inf\{|\hat{\mu}(\gamma)| : \hat{\mu}(\gamma) \ne 0\} > 0$, *then* μ *is* L^p*-improving if and only if the cardinality of the support of* $\hat{\mu}$ *is finite.*

PROOF. Sufficiency is clear. For necessity note that the hypotheses imply that $\operatorname{supp}\hat{\mu}$ is contained in a $\Lambda(p)$ set for some $p > 2$. A basic property of $\Lambda(p)$ sets is that such measures are actually L^p functions [**4**]; so $\limsup_{\gamma\in\Gamma} |\hat{\mu}(\gamma)| = 0$.

Much is known about the structure of $\Lambda(p)$ sets (cf., [**4**, Chapter 6 and **9**]). To cite but one example: it is known that if A is an arithmetic progression of length N, and E is a $\Lambda(p)$ set for some $p > 2$, then $|A \cap E| \le C\Lambda(p, E)^2 N^{2/p}$, where C is a constant independent of N, E and p [**9**, 3.5]. (Here $|\cdot|$ denotes the cardinality of the set.)

Thus if μ is an L^p-improving measure and $\|\mu\| \le 1$, then for each $0 < \varepsilon \le 1$ and $2 < p < \infty$

$$|A \cap E(\varepsilon)| \le \frac{C_1}{\varepsilon^2} p^{-2C_2 \log\varepsilon} N^{2/p},$$

where C_1 and C_2 are constants independent of p, ε and N. Taking

$$p = (-1/C_2 \log\varepsilon)\log N$$

we obtain

COROLLARY 5. *Let* μ *be an* L^p*-improving measure with* $\|\mu\| \le 1$. *There are constants* C_1 *and* C_2, *independent of* N, *so that if* A *is an arithmetic progression of length* N, *then*

$$|A \cap E(\varepsilon)| \le C_1(\log N)^{-2C_2 \log\varepsilon}.$$

A measure μ, acting as a convolution operator from L^1 to L^1, is said to be an *Enflo operator* if there is a subspace Y of L^1, isomorphic to L^1, on which μ is an isomorphism. In [**8**] Rosenthal proves that if for each $\varepsilon > 0$, $\{\gamma : |\hat{\mu}(\gamma)| > \varepsilon\}$ is a $\Lambda(p)$ set for some $p > 2$, then the measure μ is non-Enflo. Consequently, all L^p-improving measures are non-Enflo.

We will say that a measure μ has property $(*)$ if whenever R is an infinite dimensional reflexive subspace of L^1, and $\mu|_R$ is an isomorphism onto its range, then R is isomorphic to a Hilbert space. Rosenthal asks in [**8**] if there are any measures μ which have property $(*)$ and for which $\limsup_{\gamma\in\Gamma} |\hat{\mu}(\gamma)| \ne 0$. Our final proposition answers this question affirmatively.

PROPOSITION. *Any L^p-improving measure has property* $(*)$.

REMARKS 1. This is a generalization of the fact that the $E(\varepsilon)$ sets for an L^p-improving measure are $\Lambda(p)$ for all $p < \infty$, so that $L^p_{E(\varepsilon)} \cong L^2$.

2. Rosenthal has communicated to us that Bourgain, in an unpublished proof, showed that Riesz products have property $(*)$.

PROOF. Let μ be an L^p-improving measure and R an infinite dimensional reflexive subspace of L^1. Then R is isomorphic to a subspace of L^p for some $p > 1$ [**7**]. Fix $1 < r < p$ and choose $R_1 \cong R$ as in [**8**] so that $R_1 \subseteq L^r$ and $\mu|_{R_1}$ is an isomorphism onto its range. R is closed in L^1; hence R_1 is closed in L^r and $\mu * R_1$ is a closed subspace of L^1.

Since μ is L^p-improving there is a constant $\delta > 0$ so that $\mu * L^r \subseteq L^{r+\delta}$. In particular, $\mu * R_1 \subseteq L^{r+\delta}$. Since $\mu * R_1$ is closed in L^1, it is closed in $L^{r+\delta}$, and thus R is isomorphic to a closed subspace of $L^{r+\delta}$.

Fix r_1, with $r < r_1 < r + \delta$ and let $s = (r+\delta)/r$. Let $r(n) = r_1^n/r^{n-1}$. The complex interpolation method shows that $\mu * L^{r(n)} \subseteq L^{r(n)s}$ for all $n \geq 0$. Suppose we inductively assume that $R \cong R_{n+1}$, a closed subspace of $L^{r(n)}$, and $\mu|_{R_{n+1}}$ is an isomorphism onto its range. Then $R \cong \mu * R_{n+1}$, a closed subspace of $L^{r(n)s}$, and since $r(n+1) < r(n)s$, there is a closed subspace R_{n+2} of $L^{r(n+1)}$ isomorphic to R on which μ is an isomorphism. If n is chosen so that $r(n) \geq 2$, then R is isomorphic to $\mu * R_{n+1}$, a closed subspace of L^2, proving the result.

In conclusion, we would like to thank C. Graham for introducing us to the notion of L^p-improving measures.

BIBLIOGRAPHY

1. W. Beckner, S. Janson and D. Jerison, *Convolution inequalities on the circle*, Conference on Harmonic Analysis in Honor of Antoni Zygmund (W. Beckner et al., eds.), Wadsworth, Belmont, Calif., 1983.
2. A. Bonami, *Etude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) **20** (1970), 335–402.
3. M. Christ, *A convolution inequality concerning Cantor-Lebesgue measures*, preprint, 1985.
4. J. Lopez and K. Ross, *Sidon sets*, Lecture Notes in Pure and Appl. Math., no. 13, Marcel Dekker, New York, 1975.
5. D. Oberlin, *A convolution property of the Cantor-Lebesgue measure*, Colloq. Math. **67** (1982), 113–117.
6. D. Ritter, *Most Riesz product measures are L^p-improving*, Proc. Amer. Math. Soc. **97** (1986), 291–295.
7. H. P. Rosenthal, *On subspaces of L^p*, Ann. of Math. **97** (1973), 344–373.
8. ——, *Convolution by a biased coin*, vol. II, Altgeld Book, Univ. of Illinois, 1975, pp. 1–17.
9. W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.
10. E. M. Stein, *Harmonic analysis on R^n*, Studies in Harmonic Analysis (J. M. Ash, ed.), M.A.A. Studies Math., vol. 13, Math. Assoc. Amer., Washington, D.C., 1976, pp. 97–135.
11. A. Zygmund, *Trigonometric series*, 2nd ed., vol. II, Cambridge Univ. Press, 1959.

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