SPECTRUM OF THE PRODUCT OF OPERATORS

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ABSTRACT. It is shown that the product of two operators on a Hilbert space
has real spectrum if one of them is symmetric and the other is positive. Also,
the product of two positive operators has positive spectrum.

In general we do not know much about the spectrum of the product of two
(bounded, linear) operators. The Jacobson Lemma which states that the two products AB and BA of operators A and B have the same spectrum except possibly
for the point zero, is one of the few known facts. In finite dimensions there are
results under additional hypotheses that, say, operators A and B are orthogonal
projections [1] (see also [3]) or, that they are strictly positive [2]. Here, we generalize a finite-dimensional result of Hu-yun [2] on the spectrum of the product of two
positive operators.

Our first result, Proposition 1, is an immediate consequence of the following
obvious facts: (1) If B is a surjective operator on a Hilbert space H such that the
range of B^* is dense in H, then B is invertible. (2) If A and B are two operators
on H such that the product BA is invertible and 0 is not in the residual spectrum
of B^*, then B is invertible. Here and in the sequel A^* denotes the Hilbert space
adjoint of the operator A. Besides, "0 is in the residual spectrum of an operator
A" means that A has trivial kernel and its range is not dense in H.

PROPOSITION 1. *Let A and B be operators on a Hilbert space H, let B be
positive and denote by P the positive square root of the operator B. Then,*

$$\sigma(AB) = \sigma(BA) = \sigma(PAP).$$

PROOF. The nonzero points of the spectrum of the two products AB and BA
coincide by the Jacobson Lemma. If BA is invertible, use assertion (2) above to
get that AB is also invertible. To get the other implication, use the first one for
the adjoints. The second equality stated in the proposition follows from the fact
that $AB = (AP)P$ using the above argument for the product of AP and P.

REMARK. Observe that it can easily be proved that in the case $\operatorname{Ker} A^* =
\operatorname{Ker} A = \operatorname{Ker} B = \{0\}$, the operators AB and PAP are quasisimilar in the sense of
Sz.-Nagy and Foias [4]. However, even when A and consequently PAP are selfadjoint, the standard theory of quasisimilarity gives us only the inclusion $\sigma(PAP) \subset
\sigma(AB)$ in general.

In the following we shall need the notion of the numerical range $W(S)$ of an
operator S on H, defined as the set $\{(Sx, x); \|x\| = 1\}$ where $(\cdot, \cdot)$ denotes the

inner product of the space H. It is well known that for a normal operator S the closure of the set $W(S)$ is the closed convex hull of the spectrum $\sigma(S)$. If S is selfadjoint, denote also $m(S) = \inf W(S)$ and $M(S) = \sup W(S)$.

PROPOSITION 2. *Let A be a selfadjoint operator, let B be a positive operator, and denote by P the positive square root of B. Then*
(a) *if $m(A) < 0 < M(A)$, then $W(PAP)$ is contained in the interval $[M(B)m(A)$, $M(B)M(A)]$,*
(b) *if A is positive, then $W(PAP)$ is contained in $[m(A)m(B), M(A)M(B)]$,*
(c) *if A is negative, then $W(PAP)$ is contained in $[m(A)M(B), M(A)m(B)]$.*

PROOF. The proof of case (a) will be given only. Since $m(A) < 0 < M(A)$, it holds for any vector $x \in H$ that

$$(PAPx, x) = (APx, Px) \leq M(A)\|Px\|^2 = M(A)(Px, Px)$$
$$= M(A)(Bx, x) \leq M(A)M(B)\|x\|^2$$

and also

$$(PAPx, x) = (APx, Px) \geq m(A)\|Px\|^2 = m(A)(Px, Px)$$
$$= m(A)(Bx, x) \geq m(A)M(B)\|x\|^2.$$

Both propositions together give us an interval in which the spectra of the products AB and BA of a selfadjoint operator A and a positive operator B are contained. The endpoints of this interval may or may not belong to the spectrum of the product AB, as it can be seen by taking appropriate multiplication operators on the space $L^2[0,1]$.

Recall now the following result of Hu-yun [2]. Denote by μ_i, ν_i, and λ_i respectively the eigenvalues of the $n \times n$ matrices A, B, and AB where A and B are strictly positive. Then,

$$\frac{2}{n} \frac{(\min \mu_i)^2 (\min \nu_i)^2}{(\min \mu_i)^2 + (\min \nu_i)^2} < \lambda_i < \frac{n}{2}[(\max \mu_i)^2 + (\max \nu_i)^2].$$

Our Proposition 2(b) gives another estimate

$$(\min \mu_i)(\min \nu_i) \leq \lambda_i \leq (\max \mu_i)(\max \nu_i)$$

which is sharper even in the case $n = 2$.

We have thus seen that the spectrum of the product of a selfadjoint and a positive operator is always real. A natural question arises whether the product of two selfadjoint operators has always real spectrum. The answer is no even in two dimensions. Namely, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

to get a product AB with purely imaginary spectrum.

References

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