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ON HÖRMANDER'S RATIO THEOREMS

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ABSTRACT. In this paper we prove a theorem concerning the coincidence of ranges of the ratio of abstract homogeneous polynomials and the corresponding ratio of their polars. We employ the coincidence theorem on symmetric multilinear forms (proved recently by the authors [**6** or **5**]) to prove our theorem. The main result deduces two ratio theorems due to Hörmander [**4**], one being an improvement upon his version in the complex plane.

1. Introduction. In a recent paper by the authors [**6**], a well-known classical result on symmetric n-linear forms, commonly known as Walsh's coincidence theorem (cf. [**9** or **7**, Theorem (15.3)]), was genearlized to vector-valued symmetric multilinear forms. In the same paper, this generalization was found to have varied applications and its versatility was established by deducing some well-known and seemingly unrelated results, sometimes also furnishing alternate proofs of a large number of results. In the present paper, we discover yet another instance of its applicability that further widens its spectrum. In fact, the above-mentioned result on symmetric multilinear forms is instrumental in obtaining a generalization of a result of Hörmander [**4**] on the range of certain ratios of abstract homogeneous polynomials.

Throughout the paper, K denotes an algebraically closed field of characteristic zero with a maximal ordered subfield K_0, so that $K = K_0(i) = \{a + ib\colon a, b \in K_0\}$, where $-i^2 = 1$ (cf. [**1**, **4**, **8**]). We write $K_\omega = K \cup \{\omega\}$, where ω has the properties of infinity, and denote by $D(K_\omega)$ the family of all *generalized circular regions* (briefly, *g.c.r.*) of K_ω as defined by Zervos [**12**]. A complete account with references about these and other details can be found in [**11**, pp. 527–528], including the following result due to Zervos: If $\mathbf{C}$ denotes the field of complex numbers, then *the nontrivial g.c.r.s of* $\mathbf{C}_\omega$ *are precisely those subsets of* $\mathbf{C}_\omega$ *which are bounded by a circle or a straight line and which contain a connected subset* (*possibly empty*) *of their boundary.* We say that A is a *classical c.r.* if $A \in D(\mathbf{C}_\omega)$ with all or no boundary points of A included in A.

2. Preliminaries. If E is a vector space over K, we write E^n for the set of all n-tuples of elements of E. In what follows we borrow the notation and terminology

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from [10]. Given a nucleus N ($\subseteq E^2$) and a circular mapping $G\colon N \to D(K_\omega)$, we define the *circular cone* $E_0(N;G)$ in E by

$$E_0(N;G) = \bigcup T_G(x,y), \tag{2.1}$$

where

$$T_G(x,y) = \{sx + ty \neq 0;\ s,t \in K,\ s/t \in G(x,y)\} \tag{2.2}$$

and the union in (2.1) ranges over all $(x,y) \in N$. Circular cones are very intimately related to hermitian cones as exhibited in the following propositions. (For details, see [**10**, pp. 117–118].)

PROPOSITION (2.1). *Let E_1 be a hermitian cone in E. Given a nucleus N of E^2, there exists a circular mapping $G\colon N \to D(K_\omega)$ such that the corresponding cone $E_0(N;G) = E_1$ and $E_1 \cap \mathcal{L}[x,y] = T_G(x,y)$ for all $(x,y) \in N$, where $T_G(x,y)$ is defined by* (2.2)*, and $\mathcal{L}[x,y]$ denotes the subspace of E generated by x and y.*

PROPOSITION (2.2). *The class of all circular cones in E properly contains the class of all hermitian cones in E.*

A mapping $P\colon E \to K$ is called (cf. [**10**, p. 115; **4**, pp. 55, 59; **3**, pp. 760–763]) an *abstract homogeneous polynomial* (briefly, *a.h.p.*) of degree n if for every $x, y \in E$,

$$P(sx + ty) = \sum_{k=0}^{n} A_k(x,y) s^k t^{n-k} \quad \forall s,t \in K, \tag{2.3}$$

where $A_k(x,y) \in K$ and are independent of s and t. $\mathbf{P}_n$ will denote the class of all nth degree a.h.p.s. Obviously, $P(sx) = s^n P(x)$ for $P \in \mathbf{P}_n$, $x \in E$ and $s \in K$.

A symmetric n-linear form F on E is a mapping $F(x_1, x_2, \ldots, x_n)$ from E^n to K which is symmetric in the set $\{x_1, x_2, \ldots, x_n\}$ and linear in each x_k separately. The *nth polar* of $P \in \mathbf{P}_n$ is the mapping (see [**4**, Lemma 1 or 3, p. 763] for its existence and uniqueness) $P(x_1, x_2, \ldots, x_n)$ from E^n to K which is a symmetric n-linear form such that $P(x, x, \ldots, x) = P(x)$ for all $x \in E$. We then specify the *kth polar P_k* of p by

$$P_k(x) \equiv P_k(x_1, x_2, \ldots, x_k, x) \equiv P(x_1, x_2, \ldots, x_k, x, x, \ldots, x) \tag{2.4}$$

for all $x \in E$.

REMARK (2.3). (i) Using a proof similar to the one in Hille and Phillips [**3**, p. 763], we see that if $F(x_1, x_2, \ldots, x_n)$ is a symmetric n-linear form from E^n to K, then the mapping

$$P(x) = F(x, x, \ldots, x) \quad \forall x \in E$$

defines an a.h.p. $P \in \mathbf{P}_n$, and the nth polar $P(x_1, x_2, \ldots, x_n)$ of P coincides with $F(x_1, x_2, \ldots, x_n)$ on E^n.

(ii) If the elements $x_1, x_2, \ldots, x_k \in E$ are held fixed while the remaining $(n-k)$ points $x_{k+1}, x_{k+2}, \ldots, x_n$ are variables, then

$$P(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n)$$

is a symmetric $(n-k)$-linear form from $E^{(n-k)}$ to K. Conversely, by Remark (2.3)(i), the kth polar P_k of P in (2.4) defines an a.h.p. of degree $(n-k)$. That is,

$$P \in \mathbf{P}_n \quad \text{implies that} \quad P_k \in \mathbf{P}_{n-k}.$$

The following coincidence theorem (cf. [**6**, Theorem 3.3]) on symmetric multilinear forms generalizes Walsh's coincidence theorem (cf. [**9** or **7**, Theorem (15, 4)]) and was shown [**6**, Theorem 3.4] to be equivalent to a previous result of Zaheer [**10**, Theorem (4.2)] that was obtained as a generalization of Grace's theorem (cf. [**2** or **7**, Theorem (15,3)]).

THEOREM (2.4). *Let $P(x_1, x_2, \ldots, x_n)$ be a symmetric n-linear form from E^n to K and let $E_0(N;G)$ be a circular cone in E. If $\xi_j \in T_G(x_0, y_0)$, $j = 1, 2, \ldots, n$, for some $(x_0, y_0) \in N$ such that $P(\xi_1, \xi_2, \ldots, \xi_n) = 0$, then there exists an element $\xi \in T_G(x_0, y_0)$ such that*

$$P(\xi, \xi, \ldots, \xi) = P(\xi_1, \xi_2, \ldots, \xi_n).$$

3. A generalization of Hörmander's theorem. This section concerns an application of Theorem (2.4) that leads to the following generalization of two ratio theorems of Hörmander (cf. [**4**, Theorem 3, p. 59, and Theorem 3′, p. 62]).

THEOREM (3.1). *Let $E_0(N;G)$ be a circular cone in E and $P, Q \in \mathbf{P}_n$ be a.h.p.s from E to K. If $Q(x) \neq 0$ for every $x \in T_G(x_0, y_0)$ for some $(x_0, y_0) \in N$, then the range of $P(x)/Q(x)$ with $x \in T_G(x_0, y_0)$, and the range of*

$$P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n) \tag{3.1}$$

with $x_j \in T_G(x_0, y_0)$, $j = 1, 2, \ldots, n$, are identical.

PROOF. Since $Q(x) \neq 0$ for every $x \in T_G(x_0, y_0)$, observe that $Q(x_1, x_2, \ldots, x_n) \neq 0$ for every $x_1, x_2, \ldots, x_n \in T_G(x_0, y_0)$. For, otherwise, Theorem (2.4) implies the existence of an element $\xi \in T_G(x_0, y_0)$ such that $Q(\xi) = Q(\xi, \xi, \ldots, \xi) = Q(x_1, x_2, \ldots, x_n) = 0$, contradicting the hypothesis on Q. Therefore, the quotient in (3.1) is well defined for all $x_j \in T_G(x_0, y_0)$; suppose instead this assumes a value λ for some $x_j \in T_G(x_0, y_0)$. This implies that

$$O = P(x_1, x_2, \ldots, x_n) - \lambda Q(x_1, x_2, \ldots, x_n) = R(x_1, x_2, \ldots, x_n),$$

say. Since $R(x_1, x_2, \ldots, x_n)$ is a symmetric n-linear form, by Theorem (2.4), once again

$$\begin{aligned} 0 &= R(x_1, x_2, \ldots, x_n) = R(\xi, \xi, \ldots, \xi) \\ &= P(\xi, \xi, \ldots, \xi) - \lambda Q(\xi, \xi, \ldots, \xi) = P(\xi) - \lambda Q(\xi) \end{aligned}$$

for some $\xi \in T_G(x_0, y_0)$. Therefore, $P(\xi)/Q(\xi) = \lambda$. That is, the range of the quotient in (3.1) is contained in the range of $P(x)/Q(x)$. The reverse containment is obvious due to the fact that $P(x, x, \ldots, x) = P(x)$ (same is true for Q). This completes our proof.

Though the following corollary can be proved by a direct application of Theorem (2.4), we prefer to use Theorem (3.1) instead.

COROLLARY (3.2) (HÖRMANDER [**4**, THEOREM 3, P. 59]). *Let E_1 be a hermitian cone in E and $P, Q \in \mathbf{P}_n$ be a.h.p.s from E to K. If $Q(x) \neq 0$ for every $x \in E_1$, then the range of $P(x)/Q(x)$ with $x \in E_1$ and the range of $P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n)$ with $x_j \in E_1$ are identical.*

PROOF. (a) First we prove the theorem when $x_2 = x_3 = \cdots = x_n = x$ (say). If N is any arbitrarily selected nucleus of E^2, there exists (cf. Proposition (2.1)) a circular mapping G such that $E_1 = E_0(N;G)$ and $E_1 \cap \mathcal{L}[x, y] = T_G(x, y)$ for all $(x,y) \in N$. Therefore, the hypothesis on Q says that for each $(x',y') \in N$ the value $Q(x) \neq 0$ for all $x \in T_G(x',y')$ and so (as in the proof of Theorem (3.1)) $Q(x_1,x_2,\ldots,x_n) \neq 0$ for all $x_j \in T_G(x',y')$. Let

$$P(x_1,x,x,\ldots,x)/Q(x_1,x,x,\ldots,x) = \lambda \tag{3.2}$$

for some $x_1, x \in E_1$. We then choose an element $(x_0,y_0) \in N$ such that $x_1, x \in \mathcal{L}[x_0,y_0]$ (it is always possible). That is, $x_1, x \in T_G(x_0,y_0)$. By Theorem (3.1), applied to the circular cone $E_0(N;G) = E_1$ under discussion, $P(\xi)/Q(\xi) = \lambda$ for some $\xi \in T_G(x_0,y_0)$ (i.e. $\xi \in E_1$). Hence, the range of the quotient in (3.2) with $x_1, x \in E_1$ is a subset of the range of $P(x)/Q(x)$ with $x \in E_1$. Since the reverse containment is obvious, the proof of (a) is complete.

(b) Given $x_1 \in E_1$, let

$$P_1(x) = P(x_1,x,x,\ldots,x) \quad \forall x \in E,$$
$$Q_1(x) = Q(x_1,x,x,\ldots,x) \quad \forall x \in E.$$

Then $P_1, Q_1 \in \mathbf{P}_{n-1}$ (cf. Remark(2.3)(ii)) such that $Q_1(x) \neq 0$ for all $x \in E_1$. Applying (a) to P_1 and Q_1, we see that the range of $P_1(x)/Q_1(x)$ and the range of $P_1(x_2,x,x,\ldots,x)/Q_1(x_2,x,x,\ldots,x)$ coincide. The last quotient can be written as

$$P(x_1,x_2,x,x,\ldots,x)/Q(x_1,x_2,x,x,\ldots,x).$$

Continuing in this manner, until $n-1$ applications of the result in (a), we prove the corollary.

Proposition (2.2) now reveals that our main Theorem (3.1) is a strengthened generalization of the above Corollary (3.2) due to Hörmander. Next, we consider a.h.p.s in $E = \mathbf{C}^2$ and apply Theorem (3.1) to improve upon another result of Hörmander [**4**, p. 62]. In this result, Hörmander has used a symbolic notation for a.h.p.s from $\mathbf{C}^2$ to $\mathbf{C}$, originally pointed out to him by Professor Marcel Riesz. We explain this notation below.

Every element $x \in E = \mathbf{C}^2$ has the unique representation $x = sx_0 + ty_0 \equiv (s,t)$ for some $s,t \in \mathbf{C}$, where $x_0 = (1,0)$ and $y_0 = (0,1)$. An a.h.p. $P \in \mathbf{P}_n$ from $\mathbf{C}^2$ to $\mathbf{C}$ can be written as

$$P(x) \equiv P(sx_0 + ty_0) = \sum_{k=0}^{n} C(n,k) a_k s^k t^{n-k} \quad \forall s,t \in \mathbf{C}.$$

Or, symbolically as

$$P(x) \equiv P(sx_0 + ty_0) = (s + at)^n \quad \forall s,t \in \mathbf{C}. \tag{3.3}$$

This and the following symbolic formulas should thus be interpreted by considering a as an indeterminate, next multiplying all factors together, and then replacing a^{n-k} by a_k. One can easily see that the nth polar of $P(x)$ is given in the symbolic form by

$$P(x_1,x_2,\ldots,x_n) = (s_1 + at_1)(s_2 + at_2)\cdots(s_n + at_n), \tag{3.4}$$

where $x_j = (s_j,t_j) \in \mathbf{C}^2$, because the right-hand side of (3.4) is a symmetric n-linear form such that $P(x,x,\ldots,x) = (s+at)^n = P(x)$. Then by Remark (2.3)(i), $P(x_1,x_2,\ldots,x_n)$ is the nth polar of $P(x)$. Now, we give the following improved version of Hörmander's theorem (cf. [**4**, Theorem 3′, p. 62]) in the complex plane, in the sense that we use g.c.r.s instead of the classical ones.

COROLLARY (3.3). *Let $C \in D(\mathbf{C}_\omega)$ and let $(z+a)^n$ and $(z+b)^n$ be polynomials (from* $\mathbf{C}$ *to* $\mathbf{C}$*) such that $(z+b)^n \neq 0$ for all $z \in C$. Then the range of the values of $(z+a)^n/(z+b)^n$ with $z \in C$ and the range of the values of*

$$[(z_1+a)(z_2+a)\cdots(z_n+a)]/[(z_1+b)(z_2+b)\cdots(z_n+b)] \tag{3.5}$$

with $z_j \in C$ are identical.

PROOF. Using the above explained notation, we see (cf. [**10**, Remark (2.1)]) that the set

$$E_0(N;G) = \{sx_0 + ty_0 \colon s,t \in \mathbf{C},\ s/t \in C\} = T_G(x_0,y_0)\}$$

is a circular cone in $\mathbf{C}^2$, where $N = \{(x_0,y_0)\}$ and $G(x_0,y_0) = C$. Define a.h.p.s $P, Q \in \mathbf{P}_n$ from $\mathbf{C}^2$ to $\mathbf{C}$ by

$$P(x) \equiv P(sx_0+ty_0) = (s+at)^n \quad \forall x = (s,t) \in \mathbf{C}^2,$$
$$Q(x) \equiv Q(sx_0+ty_0) = (s+bt)^n \quad \forall x = (s,t) \in \mathbf{C}^2.$$

Setting $z = s/t$, $z_j = s_j/t_j$, $x = (s,t)$, and $x_j = (s_j,t_j)$, we observe that

$$P(x) = t^n(s/t+a)^n = t^n(z+a)^n \quad \forall t \neq 0, \tag{3.6}$$
$$Q(x) = t^n(s/t+b)^n = t^n(z+b)^n \quad \forall t \neq 0, \tag{3.7}$$

and that

$$P(x_1,x_2,\ldots,x_n) = t_1t_2\cdots t_n(s_1/t_1+a)(s_2/t_2+a)\cdots(s_n/t_n+a), \tag{3.8}$$
$$Q(x_1,x_2,\ldots,x_n) = t_1t_2\cdots t_n(s_1/t_1+b)(s_2/t_2+b)\cdots(s_n/t_n+b) \tag{3.9}$$

for all $t_j \neq 0$. Let us observe that if $\omega \in C$ we must consider $z = \omega$ as a zero of $(z+b)^n$ when $b_n = 0$. Therefore, from (3.7), clearly $Q(x) \neq 0$ for all $x \in T_G(x_0,\ y_0)$ (since $(z+b)^n \neq 0$ for all $z \in C$). Then (cf. proof of Theorem (3.1)) Theorem (2.4) states that $Q(x_1,x_2,\ldots,x_n) \neq 0$ for all $x_j \in T_G(x_0,y_0)$, which implies by (3.9) that $(z_1+b)(z_2+b)\cdots(z_n+b) \neq 0$ for all $z_j \in C$. Hence the ratio in (3.5) is well defined for $z_j \in C$. Now the relations (3.6)–(3.9) and Theorem (3.1) prove the corollary.

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