SMOOTH CONVEX t-NORMS DO NOT EXIST

C. ALSINA AND M. S. TOMAS

(Communicated by Richard R. Goldberg)

ABSTRACT. We show that smooth convex t-norms do not exist.

Let $T$ be a $t$-norm, i.e., a binary operation on $[0,1]$ which is associative, commutative, nondecreasing in each place, and has 1 as a unit element. A $t$-norm $T$ is convex if, for all $a, b, c, d$ in $[0,1]$, $T$ satisfies:

$$T(\lambda a + (1 - \lambda)b, \lambda c + (1 - \lambda)d) \leq \lambda T(a, c) + (1 - \lambda)T(b, d).$$

Convex $t$-norms play a fundamental role in some studies of products of probabilistic metric spaces (see [2, 6]). An example of continuous convex $t$-norm is $W(x, y) = \max(\min(x + y - 1, 0))$ and a discontinuous example is $Z(x, y) = 0$ whenever $(x, y)$ is in $[0,1) \times [0,1)$ and $Z(x, y) = W(x, y)$ otherwise.

All other examples of continuous convex $t$-norms are closely related to $W$. Consequently, it is natural to ask whether smooth convex $t$-norms can exist. Our aim in this paper is to show that they cannot. To this end we begin with the following

**LEMMA 1.** If $T$ is a continuous convex $t$-norm, then: (i) $T \leq W$; (ii) $T$ admits a representation of the form $T(x, y) = \frac{1}{x}(-1)^{-1}(f(x) + f(y))$, where $f$ (the additive generator of $T$) is a continuous strictly decreasing function from $[0,1]$ onto $[0,1]$ such that $f(0) = 1$, $f(1) = 0$, and $f^{-1}$ is the pseudo-inverse of $f$, i.e., $f^{-1} = f^{-1}$ on $[0,1]$ and $f^{-1}(x) = 0$, whenever $x \geq 1$; (iii) $f$ is concave on $[f^{-1}(\frac{1}{2}), 1]$.

**PROOF.** Since $T$ is a convex $t$-norm, for any $a, b$ and $\lambda$ in $[0,1]$ we have

$$T(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) \leq \lambda T(a, b) + (1 - \lambda)T(b, a) = T(a, b).$$

Suppose $x, y$ in $[0,1]$ are such that $x + y > 1$. Then substituting $a = x + y - 1$, $b = 1$, and $\lambda = (1 - x)/(2 - x - y)$ in (1) yields $T(x, y) \leq T(x + y - 1, 1) = x + y - 1$, from which (i) follows. The representation of $T$ follows from the theorem of Ling [5]. Finally, the substitution $\lambda = \frac{1}{2}$ into (1) yields $T(\frac{1}{2}(a + b), \frac{1}{2}(a + b)) \leq T(a, b)$. Using the representation, for any $a, b$ in $[f^{-1}(\frac{1}{2}), 1]$ we have $(a + b)/2 \geq f^{-1}(\frac{1}{2})$ and $2f((a + b)/2) < 1$. Therefore $f^{-1}(2f((a + b)/2)) > 0$ and we obtain

$$0 < f^{-1}\left(2f\left(\frac{a + b}{2}\right)\right) = T\left(\frac{a + b}{2}, \frac{a + b}{2}\right) \leq T(a, b) = f^{-1}(f(a) + f(b)),$$

whence $f((a + b)/2) \geq \frac{1}{2}(f(a) + f(b))$, i.e., $f$ is concave on $[f^{-1}(\frac{1}{2}), 1]$.

**DEFINITION 1.** A $t$-norm $T$ representable in the form

$$T(x, y) = f^{-1}(f(x) + f(y))$$

Received by the editors April 2, 1986 and, in revised form, September 15, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 39C05, 26B25.

©1988 American Mathematical Society
0002-9939/88 $1.00 + .25 per page

317
will be called smooth if \( f' \) and \( f'' \) exist on \((0,1)\), \( f'(x) \neq 0 \) for all \( x \) in \((0,1)\), \( \lim_{x \to 0^+} x \cdot f'(x) = -\infty \), and \( f''(x) \) is bounded on an arbitrary left-neighborhood of 1 of the form \((1 - \varepsilon, 1)\).

Thus for smooth \( t \)-norms \( T \) the first and second partial derivatives of \( T \) exist on \((0,1) \times (0,1)\).

**Lemma 2.** If \( T(x,y) = f^{-1}(f(x) + f(y)) \) is a smooth convex \( t \)-norm, then

\[
\lim_{k \to 1^-} \frac{f''(k)}{f'(k)^2} = 0.
\]

**Proof.** Since \( f'' \) is assumed to be bounded on a neighborhood of 1, we will prove that \( \lim_{k \to 1^-} f'(k) = -\infty \). Choose a sequence \((y_n)_{n \in N}\) converging to 0 so that \((1 - f(y_n))/y_n > n\). (This is possible because \( \lim_{x \to 0^+} f'(x) = -\infty \).) Let \( x_n = f^{-1}(1 - f(y_n)) \). Since \( f(x_n) + f(y_n) = 1 \) and \( T \leq W \) we have

\[
0 = T(y_n, 1 - y_n) = f^{-1}(1 - f(x_n) + f(1 - y_n)),
\]

i.e., \( 1 - f(x_n) + f(1 - y_n) \geq 1 \). Hence \( 1 - x_n \leq y_n \) and we have

\[
\frac{f(x_n)}{1 - x_n} \geq \frac{f(x_n)}{y_n} = \frac{1 - f(y_n)}{y_n} > n.
\]

It follows that

\[
\lim_{x \to 1^-} f'(x) = \lim_{n \to \infty} \frac{f(x_n)}{1 - x_n} = -\infty.
\]

**Lemma 3.** If \( g \) is a function from \((0,1)\) into \( \mathbb{R} \) satisfying

(i) \( g(u) \leq 0 \) whenever \( u \) is in \((0, \frac{1}{2})\);

(ii) \( g(u) \leq g(u + v) \) for all \( u, v \) in \((0,1)\) with \( u + v \leq 1 \);

(iii) \( g(u) \cdot g(v) \geq (g(u) + g(v)) \cdot g(u + v) \) for all \( u, v \) in \((0,1)\) with \( u + v \leq 1 \);

(iv) \( \lim_{h \to 0^+} g(h) \) exists

then \( g(x) = 0 \) for all \( x \) in \((0,1)\).

**Proof.** Given any \( x \) in \((0, \frac{1}{2})\) the substitution \( u = v = x \) into (iii) yields \( g(x)^2 \geq 2g(x)g(2x) \). If \( g(x) < 0 \) then \( g(x) \leq 2g(2x) \) and by induction

\[
g(2x) \geq \frac{1}{2} g(x) \geq \frac{1}{2^{n+1}} \cdot g \left( \frac{x}{2^n} \right),
\]

whence by (iv)

\[
g(2x) \geq \frac{1}{2} g(x) \geq \lim_{n \to \infty} \frac{1}{2^{n+1}} \cdot g \left( \frac{x}{2^n} \right) = 0,
\]

i.e., \( g(x) \geq 0 \) in contradiction with \( g(x) < 0 \). Therefore by (i) we have \( g(x) = 0 \) whenever \( x \) is in \((0, \frac{1}{2})\). Now for any \( x \) in \((0,1)\) we have by (ii), \( g(x/2) = 0 \leq g(x) \). Consequently for any \( u, v \) in \((0,1)\) with \( u + v \leq 1 \) we will have by (iii) and (ii):

\[
g(u) \cdot g(v) \geq (g(u) + g(v)) \cdot g(u + v) \geq (g(u) + g(v))g(u).
\]

Thus \( 0 \geq g(u)^2 \) and we can conclude \( g(u) = 0 \) for all \( u \) in \((0,1)\). We remark that condition (iv) is essential because there are nonzero functions such as \( g(x) = -1/x \) that satisfy (i), (ii), and (iii).
THEOREM 1. Smooth convex t-norms cannot exist.

PROOF. Let T be a continuous convex t-norm. By Lemma 1, T can be represented in the form

\[ T(x, y) = f^{-1}(f(x) + f(y)) \]

where f is concave on \([f^{-1}(\frac{1}{2}), 1] \). If T is smooth and convex then it is well known [4] that the following conditions hold:

\[ \frac{\partial^2 T}{\partial x^2}(u, v) \geq 0 \]

and

\[ \frac{\partial^2 T}{\partial x^2}(u, v) \cdot \frac{\partial^2 T}{\partial y^2}(u, v) - \left[ \frac{\partial^2 T}{\partial x \partial y}(u, v) \right]^2 \geq 0 \]

for all \( u, v \) in \((0, 1)\). Using (2), after some computations we have that for any \( x, y \) in \((0, 1)\) with \( f(x) + f(y) < 1 \) the inequalities (3) and (4) become

\[ \frac{f''(x)}{f'(x)^2} \leq \frac{f''(T(x, y))}{f'(T(x, y))^2} \]

and

\[ \frac{f''(x)f''(y)}{f'(x)^2f'(y)^2} \geq \frac{f''(x)f''(T(x, y))}{f'(x)^2f'(T(x, y))^2} + \frac{f''(y)f''(T(x, y))}{f'(y)^2f'(T(x, y))^2} \]

Let \( f(x) = u \) and \( f(y) = v \). Then for \( u, v \) in \((0, 1)\) with \( u + v < 1 \), we have

\[ \frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} \leq \frac{f''(f^{-1}(u + v))}{f'(f^{-1}(u + v))^2} \]

and

\[ \frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} \cdot \frac{f''(f^{-1}(v))}{f'(f^{-1}(v))^2} \geq \frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} + \frac{f''(f^{-1}(v))}{f'(f^{-1}(v))^2} \cdot \frac{f''(f^{-1}(u + v))}{f'(f^{-1}(u + v))^2} \]

Define \( g \) from \((0, 1)\) into \( \mathbb{R} \) by \( g(z) = f''(f^{-1}(z))f'(f^{-1}(z))^2 \). Obviously \( g \) is well defined and by Lemma 2 we have

\[ \lim_{k \to 1^-} \frac{f''(k)}{f'(k)^2} = \lim_{h \to 0^+} g(h) = 0. \]

Since \( f \) is concave on \([f^{-1}(\frac{1}{2}), 1] \) we have

\[ g(u) \leq 0 \quad \text{whenever } u \text{ is in } (0, \frac{1}{2}); \]

and by (7) and (8) for all \( u, v \) in \((0, 1)\) with \( u + v < 1 \) we obtain

\[ g(u) \leq g(u + v) \]

and

\[ g(u) \cdot g(v) \geq (g(u) + g(v)) \cdot g(u + v). \]
Now we can apply Lemma 3 and conclude that $g$ must be the zero function, i.e., that $f''(f^{-1}(z)) = 0$ for all $z$ in $(0,1)$. This in turn implies that $f$ must have the form $f(x) = ax + b$ for some constants $a, b$ and contradicts the assumption that $\lim_{x \to 0^+} f'(x) = -\infty$.

ACKNOWLEDGMENTS. The authors thank Professor M. J. Frank (Illinois Institute of Technology, Chicago) for his interesting remarks concerning this paper and the referee for some helpful comments.

BIBLIOGRAPHY