PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

c_0-SUBSPACES AND FOURTH DUAL TYPES

VASILIKI A. FARMAKI

(Communicated by John B. Conway)

ABSTRACT. For a separable Banach space X, we define the notion of a c_{0^+}-type on X and show that the existence of such a type is equivalent to the embeddability of c_0 in X. All these types are weakly null and fourth dual (i.e. of the form $\tau(x) = \|x+g\|$ for $g \in X^{****}$). We define the l^{l^+}-dual types on X (these are generated by sequences in X^{**}) and prove that they coincide with the fourth dual types on X. We also prove that c_{0^+}-types are fourth dual types.

Introduction. The concept of a type on a separable Banach space X (i.e. a function of the form $\tau(x) = \lim_n \|x+x_n\|$) has proved very fruitful in the isomorphic theory of Banach spaces in recent years. For example, Maurey proved in [**4**] that a separable Banach space X contains l^1 if and only if there exists $g \in X^{**}$ such that $\|x+g\| = \|x-g\|$ for all $x \in X$. In the language of types, this means that X admits a symmetric second dual type (i.e. a type τ of the form $\tau(x) = \tau(-x) = \|x+g\|$ for some $g \in X^{**}$) as defined by Haydon and Maurey [**2**].

Maurey's results were refined by Rosenthal in [**6**], where the important class of l^{1^+}-types was introduced and shown to coincide with the class of second dual types. The existence of such types is characteristic of nonreflexive separable Banach spaces and implies that the positive face of the unit ball of l^1 embeds in X almost isometrically. Rosenthal also defined l^p-types for $1 \leq p \leq \infty$, and showed that their existence implies the almost isometric embeddability of l^p (for $p < \infty$) or c_0 (for $p = \infty$) in X. For $p = 1$ and $p = \infty$ the converse also holds [**4, 6**].

In the first part of this note we introduce the notion of a c_{0^+}-type and prove that c_{0^+}-types have properties analogous to those of the l^{1^+}-types. For example, their existence implies that the positive face of the unit ball of c_0 embeds in X almost isometrically. Also, a symmetric type is an l^∞-type if and only if it is a c_{0^+}-type, in just the same way that a symmetric type is an l^1-type if and only if it is an l^{1^+}-type [**6**].

The existence of an l^{l^+}-type on X does not imply that l^1-embeds in X. It is therefore rather striking that the existence of a nontrivial c_{0^+}-type is equivalent to the embeddability of c_0 in X (Theorem 1.8).

In the second part of this note it is shown that all c_{0^+}-type are of the form $\tau(x) = \|x+g\|$ with $g \in X^{(4)}$ ($X^{(4)}$ is the fourth dual space of X). We show that all such functions are indeed types, and we call them fourth dual types.

Received by the editors July 7, 1986 and, in revised form, October 9, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B25, 46B10.
Key words and phrases. Types on Banach spaces, c_{0^+}-types, weakly null types, l^{1^+}-dual types, fourth dual types, generation of types, isomorphic embedding.

Limits of sequences of second-dual types are themselves types, called dually generated. Within this class we define l^{1^+}-dual types, analogously to Rosenthal's l^{1^+}-types and show (Theorem 2.6) that l^{1^+}-dual types are precisely the fourth dual types. Using this theorem, we are able to characterize fourth dual types of the form $\tau(x) = \|x + g\|$ where g is a Baire-1 element of $X^{(4)}$.

The class of fourth dual types provides interesting questions for further study.

Throughout this article, we denote by X a real *separable* infinite dimensional Banach space. X^{**}, $X^{(3)}$ and $X^{(4)}$ are the second, third and fourth duals of X, respectively. For a subset A of X, $\operatorname{conv} A$ denotes the convex hull of A and $\overline{\operatorname{conv}}\, A$ the closure of $\operatorname{conv} A$. Also $[A]$ denotes the closed linear span of A.

1. c_{0^+}-types and c_0-subspaces.

DEFINITIONS 1.1. A function $\tau: X \to \mathbf{R}$ is a *type* on X if there exists a sequence (x_n) in X such that

$$\tau(x) = \lim_n \|x + x_n\| \quad \text{for } x \in X. \tag{1}$$

If (x_n) satisfies (1) we say that (x_n) *generates* τ.

A type τ on X is:

trivial if there is a $y \in X$ with $\tau(x) = \|x + y\|$ for $x \in X$,

normalized if $\tau(0) = 1$,

symmetric if $\tau(x) = \tau(-x)$ for $x \in X$,

an l^p-type for $1 \leq p \leq \infty$ if there exists a sequence (x_n) in X such that

$$\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\|$$

for $x \in X$ and $a, b \in \mathbf{R}$ with $(|a|^p + |b|^p)^{1/p} = 1$ (where we set $(|a|^\infty + |b|^\infty)^{1/\infty} = \max(|a|, |b|)$) and

an l^{1^+}-type if there exists a sequence (x_n) in X such that

$$\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\|$$

for $x \in X$ and $a, b \geq 0$ with $a + b = 1$.

These definitions follow Rosenthal [**6**], where we refer the reader for more details.

We introduce the following

DEFINITION 1.2. A function $\tau: X \to \mathbf{R}$ is a c_{0^+}-*type* if there exists a sequence (x_n) in X such that

$$\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\| \tag{2}$$

for $x \in y$ and $a, b \geq 0$ with $\max(a, b) = 1$.

If (x_n) satisfies (2), we say that (x_n) c_{0^+}-*doubly* generates τ.

Obviously a c_{0^+}-type is a type (take $a = 1$ and $b = 0$) and an l^∞-type is a c_{0^+}-type. Moreover the l^∞-types are precisely the symmetric c_{0^+}-types (the proof is similar to the proof of Proposition 1.5 in [**6**].)

DEFINITION 1.3. (a) Let $A \subset X$. An element $x \in X$ is a c_{0^+}-*convex combination* of members of A if there exist $n \in \mathbf{N}$, nonnegative scalars $\lambda_1, \ldots, \lambda_n$ with $\max(\lambda_1, \ldots, \lambda_n) = 1$ and $x_1, \ldots, x_n \in A$ such that $x = \sum_{i=1}^n \lambda_i x_i$. The set of all c_{0^+}-convex combinations of members of A is denoted by c_{0^+}-$\operatorname{conv} A$.

(b) Let (x_n) and (y_m) be sequences in X. (y_m) is called a sequence *of far-out c_{0^+}-convex combinations* of (x_n) if for every $l \in \mathbf{N}$ there exists a $k \in \mathbf{N}$ such that

$$y_m \in c_{0^+}\text{-conv}\{x_n : n \geq l\} \quad \text{for every } m \geq k.$$

DEFINITION 1.4. Let τ be a type on X. A sequence (x_n) in X *c_{0^+}-strongly generates τ* if every sequence (y_m) of far-out c_{0^+}-convex combinations of (x_n) generates τ.

The following theorem is analogous to Theorem 1.6 of **[6]** which deals with l^{1^+}-types. We write $a \sim_\varepsilon b$ for nonnegative a, b and $\varepsilon > 0$ if $|a - b| < \varepsilon$.

THEOREM 1.5. *Let τ be a type on X and (x_n) a sequence in X.*

(a) *The sequence (x_n) c_{0^+}-strongly generates τ if and only if for every $\varepsilon < 0$ and $x \in X$ there is an $l \in \mathbf{N}$ such that*

$$(3) \qquad \big| \|x + y\| - \tau(x) \big| < \varepsilon \quad \textit{for all } y \in c_{0^+}\text{-conv}\{x_n : n \geq l\}.$$

(b) *If (x_n) c_{0^+}-strongly generates τ, then τ is a c_{0^+}-type on X and (x_n) c_{0^+}-doubly generates τ.*

(c) *If (x_n) c_{0^+}-doubly generates τ, then there exists a subsequence (y_m) of (x_n) which c_{0^+}-strongly generates τ.*

PROOF. This theorem can be proved using methods analogous to Rosenthal's. We give a shorter proof (of (c)) using techniques of Maurey in **[4]**.

(c) According to Definition 1.2, we have $\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\|$ for $x \in X$ and $a, b \geq 0$ with $\max(a, b) = 1$, or equivalently

$$(4) \quad \lim_n \lim_m \|x + ax_n + bx_m\| = \lim_n \|x + \max(a, b)x_n\| \quad \text{for } x \in X \text{ and } a, b \geq 0.$$

Since X is separable, we can find an increasing sequence (F_m) of finite subsets of X with $X = \overline{\bigcup_m F_m}$. Let (ε_i) be a sequence of real positive numbers so that $\sum_{i=m}^{\infty} \varepsilon_i < 1/2^m$. We can assume that $\|x_n\| = 1$ for all $n \in \mathbf{N}$.

Using Ascoli's theorem, we select inductively a subsequence (y_m) of (x_n) so that

$$\lim_n \|x + ay_m + bx_n\| \sim_{\varepsilon_m} \lim_n \|x + \max(a, b)x_n\|$$

for all $x \in F_m + m$. Ball $[y_1, y_2, \ldots, y_{m-1}]$ and $0 \leq a,\ b \leq 1$.

Let $\varepsilon > 0$ and $x \in X$. If $x \in F_{m_0}$, choose $l \in \mathbf{N}$ such that $1/2^l < \varepsilon$ and $m_0 < l$. We claim that,

$$(5) \qquad \big| \|x + y\| - \tau(x) \big| < \varepsilon \quad \text{for } y \in c_0\text{-conv}\{y_m : m \geq l\}.$$

Indeed, let $y = \sum_{i=l}^{N} \lambda_i y_i$ with $\lambda_l, \lambda_{l+1}, \ldots, \lambda_N \geq 0$ and $\max(\lambda_l, \ldots, \lambda_N) = 1$. We have that

$$\left\| x + \sum_{i=l}^{N} \lambda_i y_i \right\| \sim_{\varepsilon_N} \lim_n \left\| x + \sum_{i=l}^{N-1} \lambda_i y_i + \lambda_N x_n \right\|$$
$$\sim_{\varepsilon_N + \varepsilon_{N-1}} \lim_n \left\| x + \sum_{i=l}^{N-2} \lambda_i y_i + \max(\lambda_N, \lambda_{N-1}) x_n \right\|.$$

Finally

$$\left\| x + \sum_{i=l}^{N} \lambda_i y_i \right\| \sim_{\sum_{i=l}^{N} \varepsilon_i} \lim_n \|x + \max(\lambda_l, \ldots, \lambda_N) x_n\| = \tau(x).$$

Thus the claim is true. Hence, by continuity, (5) holds for $x \in X$. By (a), it follows that (y_m) c_{0^+}-strongly generates τ.

DEFINITION 1.6 [1]. A type τ on X is called *weakly null* if there exists a sequence (x_n) in X such that $w\text{-}\lim_n x_n = 0$ and (x_n) generate τ.

Using Theorem 1.5 we prove that every c_{0^+}-type is weakly null.

PROPOSITION 1.7. *Every c_{0^+}-type on X is weakly null. Moreover, for each sequence (x_n) in X, which c_{0^+}-strongly generates a type τ on X, we have w-$\lim_n x_n = 0$.*

PROOF. Let τ be a c_{0^+}-type on X and (x_n) a sequence in X which c_{0^+}-strongly generates τ. According to Theorem 1.5(a) for $x = 0$ and $\varepsilon = 1$, there exists an $l \in \mathbf{N}$ such that

$$\|y\| \leq 1 + \tau(0) \quad \text{for every } y \in c_{0^+}\text{-conv}\{x_n : n \geq l\}. \tag{6}$$

Hence for each increasing sequence (n_k) of natural numbers with $n_k \geq l$ for $k \in \mathbf{N}$, we have

$$\lim_k \frac{\|x_{n_1} + \cdots + x_{n_k}\|}{k} = 0.$$

Thus the sequence (x_n) is weakly null.

Using all previous results we now give a criterion for the embeddability of c_0 in a separable Banach space.

THEOREM 1.8. *Let X be a separable Banach space. Then c_0 embeds in X if and only if there exists a nontrivial c_{0^+}-type on X.*

PROOF. If c_0 embeds in X, then there exists a nontrivial l^∞-type on X, as Rosenthal has proved in [6]. But every l^∞-type is a c_{0^+}-type.

Conversely, let τ be a nontrivial c_{0^+}-type on X and $\tau(0) = 1$. According to Theorem 1.5(c) there exists a sequence (x_n) in X which c_{0^+}-strongly generates τ. This sequence is weakly null by Proposition 1.7. We can assume that $\|x_n\| = 1$ for all $n \in \mathbf{N}$. Also, according to Theorem 1.5(a) (for $x = 0$ and $\varepsilon = 1$), there exists an $l \in \mathbf{N}$ such that

$$\|y\| \leq 2 \quad \text{for every } y \in c_{0^+}\text{-conv}\{x_n : n \geq l\}. \tag{7}$$

In particular

$$\left\| \sum_{n \in F} x_n \right\| \leq 2 \quad \text{for all finite subsets } F \text{ of } \{n \in \mathbf{N} : n \geq l\}. \tag{8}$$

Thus there exists a subsequence (y_m) of (x_n) which is basic, normalized and satisfies $\|\sum_{m \in F} y_m\| \leq 2$ for all finite $F \subseteq \mathbf{N}$; then (y_m) is equivalent to the unit vector basis of c_0.

2. Fourth dual types. In this section X denotes a separable, nonreflexive infinite-dimensional real Banach space. The space X is identified (via the canonical embedding) with a subspace of X^{**} and X^{**} with a subspace of $X^{(4)}$ (the fourth dual of X).

DEFINITION 2.1. Let $\tau\colon X \to \mathbf{R}$ be a function. A sequence (x_n^{**}) in X^{**} *dually generates* τ if

$$\tau(x) = \lim_n \|x + x_n^{**}\| \quad \text{for every } x \in X. \tag{9}$$

If (x_n^{**}) dually generates a function τ, then τ is a type on X. Indeed, for every $n \in \mathbf{N}$ the functions $\tau_n: X \to \mathbf{R}$ with $\tau_n(x) = \|x + x_n^{**}\|$ are types on X (in fact l^{1^+}-types, cf. [**6**]) and $\lim_n \tau_n(x) = \tau(x)$ for $x \in X$. But it is known that the set $T(X)$ of all types on X is a closed subset of $\mathbf{R}^X$ with respect to the topology of pointwise convergence [**3**].

DEFINITION 2.2. Let $\tau: X \to \mathbf{R}$ be a function. A sequence (x_n^{**}) in X^{**} *strongly dually generates* τ if every sequence (y_m^{**}) of far-out convex combinations of (x_n^{**}) [**6**, Definition 1.5] dually generates τ.

If a sequence in X^{**} strongly-dually generates a function τ, then obviously τ is a type on X; τ is called an l^{1^+}*-dual type.*

PROPOSITION 2.3. *Let τ be a type on X and (x_n^{**}) a sequence in X^{**}. Then (x_n^{**}) strongly dually generates τ if and only if for every $\varepsilon > 0$ and $x \in X$ there is an $l \in \mathbf{N}$ such that*

$$(10) \qquad |\|x + y^{**}\| - \tau(x)| < \varepsilon \quad \textit{for every } y \in \overline{\operatorname{conv}}\{x_n^{**} : n \geq l\}.$$

PROOF. If (x_n^{**}) does not satisfy (10) for some $\varepsilon < 0$, $x \in X$ and all $l \in \mathbf{N}$, then we can choose inductively a sequence (y_m^{**}) of far-out convex combinations of (x_n^{**}), so that (y_m^{**}) does not dually generate τ. Hence (x_n^{**}) does not strongly generate τ. The converse is immediate.

DEFINITION 2.4. A function $\tau: X \to \mathbf{R}$ is called a *fourth-dual type* on X if there exists a $g \in X^{(4)}$, so that

$$(11) \qquad \tau(x) = \|x + g\| \quad \text{for every } x \in X.$$

In this case we write $\tau = \tau_g$.

It is not immediate that a fourth dual type is a type on X. In fact, as we prove in Theorem 2.6, the fourth dual types are precisely the l^{1^+}-dual types on X, so they are types on X. The proof of Theorem 2.6 will use a concrete construction of a strongly dually generating sequences for τ_g. What is more, this sequence can be obtained from any given sequence of convex bounded sets W_n, $n \in \mathbf{N}$, in X^{**} such that $g \in \tilde{W}_n$ for every $n \in \mathbf{N}$, as described in Theorem 2.5 below. For any subset A of X^{**} (or of $X^{(4)}$) $\tilde{A}$ denotes the weak* closure of A in $X^{(4)}$.

THEOREM 2.5. *Let $g \in X^{(4)}$ and $W_1 \supset W_2 \supset \cdots$ be bounded convex subsets of X^{**} such that $g \in \bigcap_{n=1}^{\infty} \tilde{W}_n$. Then there exist convex subsets $L_1 \supset L_2 \supset \cdots$ of X^{**} so that*

(a) *$W_n \supset L_n$ for every $n \in \mathbf{N}$, and*

(b) *if $x_n^{**} \in L_n$ for every $n \in \mathbf{N}$, then the sequence (x_n^{**}) strongly dually generates τ_g.*

PROOF. We may assume that $g \notin X$. Let F, H be finite subsets of X and $X^{(3)}$ respectively, $\varepsilon > 0$ and L a bounded and convex subset of X^{**}. If

$$L_{F,H,\varepsilon}^g = \{l \in L: \|l+x\| < (1+\varepsilon)\|g+x\| \text{ for all } x \in F \text{ and } g(f)-\varepsilon < f(l) \text{ for } f \in H\}$$

then $L_{F,H,\varepsilon}^g$ is a convex subset of L and if $g \in \tilde{L}$ then $g \in \tilde{L}_{F,H,\varepsilon}^g$ (Lemma 2.5 in [**6**]).

Let $\{x_1, x_2, \ldots\}$ be a dense, countable subset of X and $F_n = \{x_1, x_2, \ldots, x_n\}$ for every $n \in \mathbf{N}$. We choose finite subsets $H_1 \subset H_2 \subset \cdots$ of the unit ball of $X^{(3)}$

such that

$$(12) \qquad \max_{f\in H_n}(g+x)(f) > \|g+x\| - 1/n \quad \text{for } x \in F_n.$$

Set $L_n = W^g_{n_{F_n,H_n,1/n}}$ for every $n \in \mathbf{N}$. The sets L_n are convex, decreasing and satisfy (a). We prove that they also satisfy (b). For each $n \in \mathbf{N}$ choose $x_n^{**} \in L_n$. If $x = x_{n_0}$ ($n_0 \in \mathbf{N}$) then

$$(13) \qquad \|x_n^{**} + x\| < (1+1/n)\|g+x\| \quad \text{for every } n \geq n_0$$

because $x \in F_n$ for every $n \geq n_0$. On the other hand, for every $n \in \mathbf{N}$, $f \in H_n$ and $x \in F_n$

$$(14) \qquad f(x_n^{**} + x) > g(f) - 1/n + f(x) = (g+x)(f) - 1/n.$$

Hence for $x = x_{n_0}$ we have

$$(15) \quad \|x_n^{**} + x\| \geq \max_{f\in H_n} f(x_n^{**}+x) > \max_{f\in H_n}(g+x)(f) - 1/n > \|g+x\| - 1/2n.$$

Relation (15), combined with (13), shows that

$$(16) \qquad \tau_g(x) = \lim_n \|x + x_n^{**}\| \quad \text{for every } x \in \{x_1, x_2, \dots\}.$$

Since $\{x_1, x_2, \dots\}$ is dense in X

$$(17) \qquad \tau_g(x) = \lim_n \|x + x_n^{**}\| \quad \text{for every } x \in X.$$

Hence (x_n^{**}) dually generates τ_g and since the sets L_n ($n \in \mathbf{N}$) are convex and decreasing, it follows that (x_n^{**}) strongly dually generates τ_g.

THEOREM 2.6. *Let X be a separable Banach space. A function $\tau\colon X \to \mathbf{R}$ is a fourth dual type on X if and only if it is an l^{1^+}-dual type on X.*

PROOF. Let $\tau = \tau_g$ be a fourth dual type on X. Using Theorem 2.5 for $W_n = \{x \in X^{**}\colon \|x\| \leq \|g\|\}$ ($n \in \mathbf{N}$) we can find a sequence (x_n^{**}) in X^{**} which strongly dually generates τ. Conversely, if τ is an l^{1^+}-dual on X and $(x_n^{**}) \subset X^{**}$ strongly dually generates τ, then using again Theorem 2.5 for $W_n = \operatorname{conv}\{x_i^{**}\colon i \geq n\}$ and $g \in \bigcap_{n=1}^\infty \tilde{W}_n$, we find a sequence (y_n^{**}) of far-out convex combinations of (x_n^{**}) which strongly dually generates τ_g. Thus $\tau = \tau_g$.

The proof of the following corollary is entirely analogous to Corollary 2.2 in [**6**], making use of Theorem 2.5.

COROLLARY 2.7. *Let τ be a type on X, (x_n^{**}) a sequence in X^{**} dually generating τ and $W_n = \operatorname{conv}\{x_i^{**}\colon i \geq n\}$ for every $n \in \mathbf{N}$. Then (x_n^{**}) strongly dually generates τ if and only if $\tau = \tau_g$ for $g \in \bigcap_{n=1}^\infty \tilde{W}_n$.*

Using Theorems 2.5 and 2.6 we give a characterization of a fourth dual type τ_g, where g is a Baire-1 element of $X^{(4)}$, i.e. a weak* limit in $X^{(4)}$ of a sequence in X^{**}. Similar results for double dual types are found in [**6**].

THEOREM 2.8. *Let τ be a type on X. There exists a Baire-1 element g of $X^{(4)}$ such that $\tau = \tau_g$ if and only if there exists a sequence (x_n^{**}) in X^{**} which is weakly Cauchy and strongly dually generates τ.*

PROOF. Let $\tau = \tau_g$ for some $g \in X^{(4)}$ and (x_n^{**}) a sequence in X^*, weak*-converging in $X^{(4)}$ to g. Using Theorem 2.5 for $W_n = \operatorname{conv}\{x_i^{**}\colon i \geq n\}$ and

$g \in \bigcap_{n=1}^{\infty} \tilde{W}_n$ we find a sequence (y_n^{**}) of far-out convex combinations of (x_n^{**}) which strongly dually generates τ. Moreover (y_n^{**}) converges weak* in $X^{(4)}$ to g. For the converse, let (x_n^{**}) be a sequence in X^{**} which strongly dually generates τ, and is weakly Cauchy. If $g \in X^{(4)}$ is the weak* limit of (x_n^{**}) in $X^{(4)}$, and $W_n = \operatorname{conv}\{x^{**} : i \geq n\}$, then $g \in \bigcap_{n=1}^{\infty} \tilde{W}_n$, and hence by Theorem 2.5, $\tau = \tau_g$.

From the previous theorem and the fact that, if a sequence has no weak-Cauchy subsequence, it has a subsequence equivalent to the usual basis of l^1 [**5**], we have

COROLLARY 2.9. *Let X be a separable Banach space. If l^1 does not embed in X^{**}, then for every l^{1^+}-dual type τ on X there is a Baire-1 element g of $X^{(4)}$ with $\tau = \tau_g$.*

Finally, using Theorem 2.6, we shall prove that every c_{0^+}-type is a fourth dual type on a separable Banach space.

PROPOSITION 2.10. *Every c_{0^+}-type on X is a fourth dual type on X. Moreover if $(x_n) \subset X$ c_{0^+}-strongly generates τ, then the sequence (y_n^{**}) in X^{**} with $y_n^{**} = w^*\text{-}\lim_{m,U}(x_n + \cdots + x_m)$ (U any nontrivial ultrafilter on $\mathbf{N}$) strongly dually generates τ.*

PROOF. Let (x_n) be a sequence in X which c_{0^+}-strongly generates τ. According to Proposition 2.3 it is sufficent to prove that for every $\varepsilon > 0$ and $x \in X$ there is an $l \in \mathbf{N}$ such that

$$\big| \|x + y^{**}\| - \tau(x) \big| < \varepsilon \quad \text{for every } y^{**} \in \overline{\operatorname{conv}}\{y_n^{**} : n \geq l\}. \tag{18}$$

Since (x_n) c_{0^+}-strongly generates τ, Theorem 1.5(a) implies that for $x \in X$ and $\varepsilon > 0$ there exists an $l \in \mathbf{N}$ such that

$$\big| \|x + y\| - \tau(x) \big| < \varepsilon \quad \text{for every } y \in \overline{c_{0^+}\text{-}\operatorname{conv}}\{x_n : n \geq l\}. \tag{19}$$

Take $y_{nm} = x_n + \cdots + x_m$ for $n, m \in \mathbf{N}$. For all natural numbers $n_k \geq \cdots \geq n_1 \geq l$ and scalars $a_1, \ldots, a_k \geq 0$ with $a_1 + a_2 + \cdots + a_k = 1$, we have

$$\operatorname{conv}\{a_1 y_{n_1 m} + \cdots + a_k y_{n_k m} : m \geq n_k\} \subset c_{0^+}\text{-}\operatorname{conv}\{x_n : n \geq l\}. \tag{20}$$

If $y^{**} \in \operatorname{conv}\{y_n^{**} : n \geq l\}$ then $y^{**} = w^*\text{-}\lim_{m,U} a_1 y_{n_1 m} + \cdots + a_k y_{n_k m}$, where $l \leq n_1 \leq \cdots \leq n_k$ and $a_1, \ldots, a_k \geq 0$ with $a_1 + \cdots + a_k = 1$. Relations (20) and (19) imply that

$$\|x + y^{**}\| \leq \lim_{m,U} \|x + a_1 y_{n_1 m} + \cdots + a_k y_{n_k m}\| \leq \varepsilon + \tau(x); \tag{21}$$

on the other hand, using the Hahn Banach theorem

$$\tau(x) - \varepsilon \leq \|x + y^{**}\|. \tag{22}$$

Thus (18) follows, and hence τ is an l^{1^+}-dual type on X. According to Theorem 2.6, τ is a fourth dual type on X.

ACKNOWLEDGMENT. The author wishes to thank Professor S. Negrepontis for his very helpful comments and suggestions.

BIBLIOGRAPHY

1. S. Argyros, S. Negrepontis and Th. Zachariades, *Weakly stable Banach spaces*, Israel J. Math. **57** (1987), 68–88.
2. R. Haydon and B. Maurey, *On Banach spaces with strongly separable types*, J. London Math. Soc. **33** (1986), 484–498.
3. J. L. Krivine and B. Maurey, *Espaces de Banach stables*, Israel J. Math. **39** (1981), 273–295.
4. B. Maurey, *Types and l^1-subspaces*, Longhorn Notes, The University of Texas, Functional Analysis Seminar, 1982–83, pp. 123–137.
5. H. Rosenthal, *A characterization of Banach space containing l^1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.
6. ——, *Double dual types and the Maurey characterization of Banach spaces containing l^1*, Longhorn Notes, The University of Texas, Functional Analysis Seminar, 1983–84, pp. 1–36.

DEPARTMENT OF MATHEMATICS, SECTION OF MATHEMATICAL ANALYSIS AND ITS APPLICATIONS, UNIVERSITY OF ATHENS, PANEPISTEMIOPOLIS, 157 81 ATHENS, GREECE