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A COUNTEREXAMPLE TO AN F. AND M. RIESZ-TYPE THEOREM

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ABSTRACT. A premeasure is a finitely additive complex-valued function μ defined on the semiring of all connected subsets of $\mathbf{T}$, continuous at $\varnothing$ and with $\mu(\varnothing) = \mu(\mathbf{T}) = 0$. Let κ be a continuous increasing concave function on $[0, 2\pi]$ with $\kappa(0) = 0$. A conjecture from [3] saying that if the Poisson integral of a premeasure μ is holomorphic in the open unit disk and $\operatorname{Var}_\kappa(\mu) < \infty$ then $\lim_{\tau\to 0} \operatorname{Var}_\kappa(\mu_\tau - \mu) = 0$ is disproved, where $\operatorname{Var}_\kappa(\mu) = \sup \sum_j |\mu(I_j)| / \sum_j \kappa(|I_j|)$ (the supremum is taken over all finite partitions of $\mathbf{T}$ into connected subsets I_j) and μ_τ denotes the τ-translation of μ.

In this paper a counterexample to a conjecture of B. Korenblum is given.

Following [2] and [3] we introduce the following notations.

$\mathbf{D}$ denotes the open unit disk in the complex plane and $\mathbf{T}$ is the unit circle. $\mathcal{I}$ denotes the family of all intervals in $\mathbf{T}$ (i.e. of all connected subsets of $\mathbf{T}$). A function $\mu\colon \mathcal{I} \to \mathbf{C}$ is called a premeasure if it satisfies the following conditions:

(i) $\mu(\varnothing) = \mu(\mathbf{T}) = 0$,

(ii) $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ for all $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 \in \mathcal{I}$,

(iii) if $I_1 \supseteq I_2 \supseteq \cdots$ is a nonincreasing sequence of intervals and $\bigcap I_n = \varnothing$, then $\lim_n \mu(I_n) = 0$.

κ will always denote an increasing concave continuous function on $[0, 2\pi]$ such that $\kappa(0) = 0$ and $\kappa(2\pi) = 1$. The *κ-variation* of a premeasure μ is

$$\operatorname{Var}_\kappa \mu = \sup \frac{\sum_1^n |\mu(I_j)|}{\sum_1^n \kappa(|I_j|)};$$

here $|I_j|$ denotes the length of the interval I_j, and the supremum is taken over all finite systems $\{I_j\}$ of mutually disjoint intervals I_j such that $\bigcup I_j = \mathbf{T}$. If $\operatorname{Var}_\kappa \mu < \infty$, μ will be called a premeasure of *bounded κ-variation.* Let V_κ denote the family of all premeasures of bounded κ-variation. V_κ with the norm $\operatorname{Var}_\kappa$ is a Banach space. If μ is a premeasure and τ is a real number then the τ-translation of μ is defined by

$$\mu_\tau(I) = \mu(\{\xi \in \mathbf{T} \colon e^{i\tau}\xi \in I\}), \qquad I \in \mathcal{I}.$$

A premeasure $\mu \in V_\kappa$ is called *κ-absolutely continuous* if $\lim_{\tau\to 0} \operatorname{Var}_\kappa(\mu - \mu_\tau) = 0$.

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If f is a C^1 function on $\mathbf{T}$ and μ is a premeasure then we can define the integral of f with respect to μ by the formula

$$\int_T f(\xi)\,d\mu(\xi) = \int_0^{2\pi} f(e^{it})\,d\tilde{\mu}(t);$$

here $\tilde{\mu}(t) = \mu((0,t])$. Therefore we can define the Poisson integral of μ by the formula

$$u(z) = u_\mu(z) = \int_{\mathbf{T}} \frac{1-|z|^2}{|\xi - z|^2}\,d\mu(\xi), \qquad z \in \mathbf{D}.$$

On the other hand, if v is a harmonic function on $\mathbf{D}$ with $v(0) = 0$ then a family $\{\mu^{(r)}\}_{r\in(0,1)}$ of premeasures is associated with v by the formula

$$\mu^{(r)}(I) = \frac{1}{2\pi}\int_{J_I} v(re^{it})\,dt, \qquad I \in \mathcal{I}; \tag{1}$$

here $J_I = \{t \in [0,2\pi] \colon e^{it} \in I\}$. It is not hard to see that if u is the Poisson integral of a κ-absolutely continuous premeasure μ, then

$$\operatorname{Var}_\kappa \mu^{(r)} \le \operatorname{Var}_\kappa \mu, \qquad r \in (0,1). \tag{2}$$

The following lemma was essentially proved in [**1**, §5].

LEMMA. *Let u be a harmonic function in $\mathbf{D}$ with $u(0) = 0$. Suppose that there is a constant C such that $|\mu^{(r)}(I)| \le C\kappa(|I|)$ for each $r \in (0,1)$ and each $I \in \mathcal{I}$, where $\{\mu^{(r)}\}_{r\in(0,1)}$ is the family of premeasures associated with u. Then u is the Poisson integral of some premeasure of κ-variation less than or equal to C.*

A premeasure μ is said to be *analytic* if its Poisson integral is holomorphic in D or, equivalently, if $\int \xi^n\,d\mu(\xi) = 0$, $n = 1, 2, \ldots$. It was conjectured in [**3**] that each analytic premeasure of bounded κ-variation is always κ-absolutely continuous. The following theorem shows that this is not true except for the classical case of the theorem of F. and M. Riesz (i.e. when $\lim_{s\to 0+} \kappa(s)/s < +\infty$).

THEOREM. *Let κ be an increasing concave continuous function on $[0,2\pi]$ such that $\kappa(0) = 0$, $\kappa(2\pi) = 1$ and $\lim_{s\to 0+}\kappa(s)/s = +\infty$. Then there exists an analytic premeasure of κ-bounded variation which is not κ-absolutely continuous.*

PROOF. Let κ be as in the assumption. It is not hard to see that there exist increasing sequences $\{N_n\}$ of positive integers, and $\{r_n\}$ of positive real numbers tending to 1, such that if we denote $\alpha_n = \kappa(\pi/N_n)/3$, then

(i) $\alpha_{n+1} \le \alpha_n/2$, $n = 1, 2, \ldots$,
(ii) $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \alpha_k N_k r_n^{N_k} = 0$,
(iii) $\lim_{n\to\infty} \max_t [\sum_{k=1}^{n-1} |\alpha_k N_k (e^{iN_k t} - e^{iN_k(t+\pi/N_n)})|] = 0$,
(iv) $\sum_{k=1}^{n-1} N_k\alpha_k \le \alpha_n N_n$, $n = 1, 2, \ldots$,
(v) $r_n^{N_n} \ge \frac{1}{2}$, $n = 1, 2, \ldots$.

Let $f_n(t) = \min\{\alpha_n, (\alpha_n N_n/\pi)t\}$, and let $\tilde{\kappa}(t) = \sum_{n=1}^{\infty} f_n(t)$, $t \in [0,2\pi]$. From (i) we deduce that $\tilde{\kappa}$ is a well-defined continuous concave function. It is constant on $[\pi/N_1, 2\pi]$ and linear on each of the intervals $[\pi/N_{n+1}, \pi/N_n]$, $n = 1, 2, \ldots$. Moreover, for each positive integer n we have

$$\tilde{\kappa}\left(\frac{\pi}{N_n}\right) = \sum_{k=1}^{n} \frac{\alpha_k N_k}{N_n} + \alpha_n + \sum_{k=n+1}^{\infty} \alpha_k.$$

By (iv) the first term of this sum does not exceed α_n. Neither, by (i), does the third. Therefore $\tilde{\kappa}(\pi/N_n) \leq 3\alpha_n = \kappa(\pi/N_n)$. Since κ is increasing and concave we have

$$\tilde{\kappa}(t) \leq \kappa(t), \qquad 0 \leq t \leq 2\pi. \tag{3}$$

Let $u(z) = \sum_{n=1}^{\infty} \alpha_n N_n z^{N_n}$, $z \in \mathbf{D}$. Since $\lim_{n\to\infty} r_n = 1$ and by (ii), the function u is well defined and holomorphic in $\mathbf{D}$. Let $\{\mu^{(r)}\}_{r\in(0,1)}$ be the family of premeasures associated with u, as defined in (1). For arbitrary $r \in (0,1)$, arbitrary $I \in \mathcal{I}$, and each n with $r_n \geq r$ we have

$$|\mu^{(r)}(I)| \leq \sum_{k=1}^{n} \left| \frac{1}{2\pi} \int_{J_I} \alpha_k N_k r^{N_k} e^{iN_k t}\, dt \right| + |I| \sum_{k=n+1}^{\infty} \alpha_k N_k r_n^{N_k}.$$

But

$$\left| \frac{1}{2\pi} \int_{J_I} \alpha_k N_k r_n^{N_k} e^{iN_k t}\, dt \right| \leq \min\left\{ \alpha_k, \frac{\alpha_k N_k}{\pi} |I| \right\}.$$

Hence, by the definition of f_k's and by (ii), $|\mu^{(r)}(I)| \leq \sum_{k=1}^{n} f_k(|I|) + |I| c_n$, where $\lim_{n\to\infty} c_n = 0$.

By (3), letting n tend to infinity in the above we obtain $|\mu^{(r)}(I)| \leq \kappa(|I|)$. Thus the lemma implies that the function u is the Poisson integral of some (obviously analytic) premeasure μ with $\operatorname{Var}_\kappa \mu \leq 1$.

For an arbitrary fixed n let

$$u_1(z) = \sum_{k=1}^{n-1} \alpha_k N_k z^{N_k}, \qquad u_2(z) = \alpha_n N_n z^{N_n},$$

and

$$u_3(z) = \sum_{k=n+1}^{\infty} \alpha_k N_k Z^{N_k}, \qquad z \in \mathbf{D}.$$

Let $\{\nu^{(r)}\}$, $\{\eta^{(r)}\}$, $\{\chi^{(r)}\}$ denote the associated families of premeasures corresponding to u_1, u_2, u_3 respectively. Denote $\nu^{(n)} = \nu^{(r_n)}$, $\eta^{(n)} = \eta^{(r_n)}$ and $\chi^{(n)} = \chi^{(r_n)}$. Note that

$$\begin{aligned} \operatorname{Var}_\kappa(\mu^{(r_n)} - \mu^{(r_n)}_{\pi/N_n}) \geq{}& \operatorname{Var}_\kappa(\eta^{(n)} - \eta^{(n)}_{\pi/N_n}) \\ &- \operatorname{Var}_\kappa(\nu^{(n)} - \nu^{(n)}_{\pi/N_n}) - 2\operatorname{Var}_\kappa \chi^{(n)}. \end{aligned} \tag{4}$$

Observe that for every harmonic function v on $\mathbf{D}$ with $v(0) = 0$ we have the inequality $\operatorname{Var}_\kappa \lambda^{(r)} \leq \sup\{|v(z)|\colon |z| = r\}$, where $\{\lambda^{(r)}\}_{r\in(0,1)}$ is the family of premeasures associated with v. If we apply this observation to u_3 then, by (ii), we have

$$\lim_{n\to\infty} \operatorname{Var}_\kappa \chi^{(n)} = 0. \tag{5}$$

The same observation applied to $v(z) = u_1(z) - u_1(ze^{i\pi/N_n})$ together with (iii) yield

$$\lim_{n\to\infty} (\nu^{(n)} - \nu^{(n)}_{\pi/N_n}) = 0. \tag{6}$$

Now, if we take $I_j = \{e^{it} \colon (j-1)\pi/N_n \le t \le j\pi/N_n\}$, $j = 1, 2, \ldots, 2N_n$, in the definition of κ-variation then we obtain

$$\begin{aligned}
\operatorname{Var}_\kappa(\eta^{(n)} - \eta^{(n)}_{\pi/N_n}) &\ge \frac{\sum_{j=1}^{2N_n} |\eta^{(n)}(I_j) - \eta^{(n)}_{\pi/N_n}(I_j)|}{\sum_{j=1}^{2N_n} \kappa(|I_j|)} \\
&= \frac{1}{2N_n \kappa(\pi/N_n)} \sum_{j=1}^{2N_n} \frac{\alpha_n N_n r_n^{N_n}}{2\pi} \left| \int_{(j-1)\pi/N_n}^{j\pi/N_n} (e^{iN_n t} - e^{iN_n(t+\pi/N_n)})\, dt \right| \\
&\ge \frac{1}{6},
\end{aligned}$$

where the last inequality follows by (v). This together with (6), (5), and (4) imply

$$\varliminf_{n\to\infty} \operatorname{Var}_\kappa(\mu^{(r_n)} - \mu^{(r_n)}_{\pi/N_n}) > 0. \tag{7}$$

If μ was κ-absolutely continuous then (7) and (2) would imply that

$$\varliminf_{n\to\infty} \operatorname{Var}_\kappa(\mu - \mu_{\pi/N_n}) > 0,$$

which contradicts the κ-absolute continuity of μ.

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