PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

REGULAR VARIATION IN $\mathbf{R}^k$

MARK M. MEERSCHAERT

(Communicated by R. Daniel Mauldin)

ABSTRACT. Researchers investigating certain limit theorems in probability have discovered a multivariable analogue to Karamata's theory of regularly varying functions. The method uses elements of real analysis and Lie groups to analyze the asymptotic behavior of functions and measures on $\mathbf{R}^k$. We present an account here which is independent of probabilistic considerations.

1. Introduction. A Borel measurable function $R\colon (0,\infty) \to (0,\infty)$ is said to vary regularly at infinity with index $\rho \in \mathbf{R}$ if, for all λ positive,

$$\lim_{x\to\infty} R(\lambda x)/R(x) = \lambda^\rho. \tag{1.1}$$

A regularly varying function with index zero is said to vary slowly. If R varies regularly with index ρ then we may always find L slowly varying such that $R(x) = x^\rho L(x)$, and hence a regularly varying function may be considered as a function whose asymptotic behavior is approximately that of a power function. The monograph by E. Seneta [**6**] contains a very readable exposition of the basic theory of regularly varying functions on $\mathbf{R}^1$.

Recently several authors have proposed multivariable analogues of the definition (1.1) in connection with certain limit theorems in probability. A. Stam used one version to prove a multivariable extension of the Abel-Tauber Theorem in [**7**]. A. L. Jakimiv apparently worked independently to produce a similar version in [**2**], which he applied to a problem in the theory of branching processes. The author of this paper applied Jakimiv's version to the problem of characterizing scalar-normed domains of attraction in $\mathbf{R}^k$ in [**4**]. More general domains of attraction problems were treated by the author in [**3, 5**], and by deHaan, Omey, and Resnick in [**1**]. The treatment of the more general domains of attraction problem led to the development of a more general version of the theory of regular variation in $\mathbf{R}^k$. The purpose of this paper is to provide an account of the multivariable theory of regular variation which is independent of probabilistic applications. The elegance and versatility of these results suggests that they may find application outside of probability (as has the one variable theory). In addition, we find the mathematics so beautiful that we feel other mathematicians will want to read it for its own sake.

The remainder of this paper is divided into three sections. In §2, we discuss regular variation of path functions on the general linear group $\mathrm{GL}(\mathbf{R}^k)$. In §3 we generalize (1.1) to the case of real-valued functions on $\mathbf{R}^k$. Finally in §4 we discuss regular variation of Borel measures on $\mathbf{R}^k$.

Received by the editors October 28, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26B35; Secondary 60F05.
Key words and phrases. Regular variation, Lie groups.

0002-9939/88 \$1.00 + \$.25 per page

2. Regular variation on $\mathrm{GL}(\mathbf{R}^k)$**.** In some ways it is most natural to define regular variation on Lie groups. Many results from $\mathbf{R}^1$ extend almost immediately. Perhaps more importantly, the results of this section are key to the development of a general theory of regularly varying functions in $\mathbf{R}^k$.

Let $\mathrm{GL}(\mathbf{R}^k)$ denote the set of invertible linear operators on $\mathbf{R}^k$. This set, together with the operation of composition (matrix multiplication), forms a Lie group, i.e., a group which is also a smooth real manifold such that the group operations $(A,B) \to AB$ and $A \to A^{-1}$ are infinitely differentiable. Let $\|A\|$ denote the norm of $A \in \mathrm{GL}(\mathbf{R}^k)$ defined as usual by

$$\|A\| = \sup\{\|Ax\| \colon \|x\| = 1\}. \tag{2.1}$$

In the norm topology on $\mathrm{GL}(\mathbf{R}^k)$ we have that $A_n \to A$ is equivalent to each of: (a) $A_n x \to Ax$ for all x in $\mathbf{R}^k$; (b) $A_n \to A$ uniformly on compact subsets of $\mathbf{R}^k$; and (c) the matrices corresponding to A_n with respect to a fixed basis for $\mathbf{R}^k$ converge elementwise to the matrix corresponding to A.

Suppose that $f\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ is Borel measurable. We will say that f varies regularly at infinity with index E if, for all $\lambda > 0$,

$$\lim_{r\to\infty} f(\lambda r) f(r)^{-1} = \lambda^E, \tag{2.2}$$

where E is some (possibly singular) linear operator on $\mathbf{R}^k$. Here λ^E denotes the operator $\exp(\log\lambda \cdot E)$, where exp is the exponential operator

$$\exp(A) = \sum_{n=0}^{\infty} A^n/n!. \tag{2.3}$$

If $E = 0$, we will say that f varies slowly.

Part of the justification for the definition (1.1) is that the limit term λ^ρ is completely general. That is, if we assume only that the limit exists and is positive for all $\lambda > 0$, then the limit must take the form of a power of λ. The next result states that the analogous result is true in the present case.

THEOREM 2.1. *Suppose that* $f\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ *is Borel measurable and that, for all* $\lambda > 0$, *we have*

$$\lim_{r\to\infty} f(\lambda r) f(r)^{-1} = \psi(\lambda) \in \mathrm{GL}(\mathbf{R}^k). \tag{2.4}$$

Then there exists a linear operator E *on* $\mathbf{R}^k$ *such that* $\psi(\lambda) = \lambda^E$ *for all* $\lambda > 0$.

PROOF. It follows from (2.4) that ψ is measurable and that $\psi(\lambda\mu) = \psi(\lambda)\psi(\mu)$ for all λ, μ positive. Clearly $\psi(1)$ is the identity operator, and so ψ is a Borel measurable group homomorphism from the multiplicative group $\mathbf{R}^+$ to $\mathrm{GL}(\mathbf{R}^k)$. Hence $\psi(e^x)$ is a measurable one-parameter subgroup, and it is well known that such a subgroup takes the form $\exp(xE)$ for some E. Reparametrize by $x = \log\lambda$.

Some of the properties of regularly varying functions on $\mathbf{R}^1$ which depend on the commutativity of multiplication on $\mathbf{R}^+$ do not extend to the present case. For instance, it is no longer true that the product of two regularly varying functions must vary regularly. In particular, we can no longer reduce to the study of slow variation, since we cannot in general write a regularly varying function as the product of a slowly varying function and a power function. In light of these limitations it is indeed interesting to see how complete we can be in generalizing the deepest and most fundamental results from the one variable case. The following theorem (which extends [**6**, p. 2]) is a good example.

THEOREM 2.2. *If f varies regularly, then the convergence in* (2.2) *is uniform on compact subsets of* $\{\lambda > 0\}$.

PROOF. We adapt the proof from Seneta's monograph. As noted above, we cannot reduce to the case of slow variation, which complicates the proof somewhat. It suffices to prove uniform convergence for λ in the interval $I = [a, b]$ where $a < 1/2$ and $b > 2$. Suppose not. Then for some $\varepsilon > 0$ there exists $r_n \to \infty$ and $\lambda_n \in I$ such that

$$\|f(\lambda_n r_n)f(r_n)^{-1} - \lambda_n^E\| \geq \varepsilon \tag{2.5}$$

for all n. Suppose each of ε_1 and ε_2 is a positive number. Define

$$\begin{aligned} U_n &= \{\lambda \in L\colon \|f(\lambda r_m)f(r_m)^{-1} - \lambda^E\| < \varepsilon_1 \ \forall m \geq n\}, \\ V_n &= \{\lambda \in L\colon \|f(\lambda\lambda_m r_m)f(\lambda_m r_m)^{-1} - \lambda^E\| < \varepsilon_2 \ \forall m \geq n\}, \end{aligned} \tag{2.6}$$

where $L = [1/2b, 2/a]$. As $n \to \infty$, the set U_n and V_n increase to L in view of the fact that (2.2) holds for each individual $\lambda > 0$. Let $V_n' = \lambda_n V_n$. By construction of L we must have $[1/2, 2]$ contained in $L \cap \lambda_n L$ for all n, and hence for some N we have $U_N \cap V_N' \neq \varnothing$. If $\lambda \in U_N$ and $\lambda \in V_N'$, then $(\lambda/\lambda_N) \in V_N$, and so

$$\begin{aligned} &\|f(\lambda r_N)f(r_N)^{-1} - \lambda^E\| < \varepsilon_1, \\ &\|f(\lambda r_N)f(\lambda_N r_N)^{-1} - (\lambda/\lambda_N)^E\| < \varepsilon_2. \end{aligned} \tag{2.7}$$

Let $A = f(\lambda r_N)f(r_N)^{-1}$ and $B = f(\lambda r_N)f(\lambda_N r_N)^{-1}$, so that

$$B^{-1}A = f(\lambda_N r_N)f(r_N)^{-1}.$$

By (2.7) and the triangle inequality, we obtain

$$\begin{aligned} \|B^{-1}A - \lambda_N^E\| &= \|B^{-1}A - B^{-1}\lambda^E + B^{-1}\lambda^E - \lambda_N^E\| \\ &\leq \|B^{-1}\| \cdot \|A - \lambda^E\| + \|B^{-1} - (\lambda_N/\lambda)^E\| \cdot \|\lambda^E\|; \end{aligned} \tag{2.8}$$

and now, by making an appropriate choice of $\varepsilon_1, \varepsilon_2$ in (2.6), we obtain a contradiction to (2.5). This completes the proof.

It follows easily from the Uniform Convergence Theorem for regularly varying real-valued functions that if R varies regularly with index ρ then $r^{\rho-\varepsilon} < R(r) < r^{\rho+\varepsilon}$ for all $r > 0$ sufficiently large. That is, the asymptotic behavior of R is approximately the same as that of r^ρ. The analogous result in the case of $\mathrm{GL}(\mathbf{R}^k)$-valued functions can be expressed as follows:

THEOREM 2.3. *Suppose that f varies regularly with index E. If all eigenvalues of E have real part greater than α, then*

$$r^{-\alpha}\|f(r)x\| \to \infty \tag{2.9}$$

as $r \to \infty$ for all nonzero x in $\mathbf{R}^k$. If all eigenvalues of E have real part less than β, then

$$r^{-\beta}\|f(r)x\| \to 0 \tag{2.10}$$

as $r \to \infty$ for all nonzero x in $\mathbf{R}^k$.

PROOF. The proof of both assertions is similar and so we only prove the first. Let S denote the unit sphere in $\mathbf{R}^k$ and let $m = \min\{\mathrm{Re}(\lambda)\}$, where λ ranges over the set of eigenvalues of E. For any $\alpha < m$, a computation shows that $r^{-\alpha}\|r^E x\| \to \infty$ uniformly on $x \in S$ as $r \to \infty$. Select a, b real with $\alpha < b < a < m$. For some $\lambda_0 > 1$ we have $\lambda_0^{-a}\|\lambda_0^E x\| > 1$ for all x in S and so $\|\lambda_0^E x\| > \lambda_0^a$. Now choose $\varepsilon > 0$ such that $\lambda_0^a - \varepsilon \geq \lambda_0^b$, and apply Theorem 2.2 to obtain $r_0 > 0$ such that $\|f(\lambda r)f(r)^{-1} - \lambda^E\| < \varepsilon$ whenever $1 \leq \lambda \leq \lambda_0$ and $r \geq r_0$. For such r, λ we have $\|f(\lambda r)f(r)^{-1}x\| \geq \lambda^b$ for all x in S. Now let $z = f(r_0)^{-1}x$ and $y = z/\|z\|$. For all n we have that $\|f(\lambda_0^n r_0)y\| \geq \lambda_0^{bn}/\|z\|$. It follows easily that (2.9) holds with y in place of x. Now by linearity the same holds for any cy, $c > 0$. But $y \in S$ is arbitrary and so we have (2.9) for all nonzero x.

3. Regularly varying functions on $\mathbf{R}^k$. Let $\Gamma = \mathbf{R}^k - \{0\}$ and suppose that $F \colon \Gamma \to [0, \infty)$ is Borel measurable. We will say that F varies regularly if there exist $f \colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ and $R \colon \mathbf{R}^+ \to \mathbf{R}^+$, both regularly varying, such that

$$\lim_{r\to\infty} F(f(r)^{-1}x_r)/R(r) = \varphi(x) > 0 \tag{3.1}$$

whenever $x_r \to x$ in Γ. Let E, β denote the index of f and R respectively. If all eigenvalues of E have positive real parts, then it follows from Theorem 2.3 that $\|f(r)^{-1}x_r\| \to 0$ as $r \to \infty$ whenever $x_r \to x$ in Γ. In this case, we will say that F varies regularly at zero. If all eigenvalues of E have negative real parts, then $\|f(r)^{-1}x_r\| \to \infty$ whenever $x_r \to x$ in Γ and in this case we will say F varies regularly at infinity.

LEMMA 3.1. *If F varies regularly and* (3.1) *holds, then*

$$\lambda^\beta \varphi(x) = \varphi(\lambda^{-E}x) \tag{3.2}$$

for all $\lambda > 0$ and all x in Γ.

PROOF. Fix λ, x and consider the equation

$$\frac{F(f(\lambda r)^{-1}x)}{R(\lambda r)} = \frac{F(f(r)^{-1}x_r)}{R(r)} \cdot \frac{R(r)}{R(\lambda r)}, \tag{3.3}$$

where $x_r = f(r)f(\lambda r)^{-1}x$. By taking inverses in (2.2) we see that $x_r \to \lambda^{-E}x$. Now letting $r \to \infty$ on both sides of (3.3) we obtain $\varphi(x) = \varphi(\lambda^{-E}x) \cdot \lambda^{-\beta}$ as desired.

LEMMA 3.2. *F varies regularly if and only if there exists f regularly varying and $e \in \Gamma$ such that*

$$\lim_{r\to\infty} F(f(r)^{-1}x_r)/F(f(r)^{-1}e) = \gamma(x) > 0 \tag{3.4}$$

whenever $x_r \to x$ in Γ.

PROOF. If F varies regularly we immediately obtain (3.4) with $\gamma(x) = \varphi(x)/\varphi(e)$. Conversely, suppose that (3.4) holds and define $R(r) = F(f(r)^{-1}e)$. For any $\lambda > 0$ we have $x_r = f(r)f(\lambda r)^{-1}e \to \lambda^{-E}e$ as $r \to \infty$ by (2.2). It follows that $R(\lambda r)/R(r)$ tends to the positive limit $\gamma(\lambda^{-E}e)$ as $r \to \infty$, and so R varies regularly. Then (3.1) holds with $\varphi \equiv \gamma$.

The preceding lemma gives an alternative definition of regular variation. The vector $e \in \Gamma$ is arbitrary, and affects the limit γ only in terms of a multiplicative constant. Lemma 3.1 and its proof illustrates the reason for the appearance of f^{-1} in the definitions (3.1) and (3.4). Notice that the regular variation of f does not imply that of f^{-1}, because of the lack of commutativity of multiplication in $\mathrm{GL}(\mathbf{R}^k)$.

Every real-valued regularly varying function R with index $\rho > 0$ has an asymptotic inverse R_1, i.e. a regularly varying function with index ρ^{-1} such that $R(R_1(r)) \sim r$ and $R_1(R(r)) \sim r$ as $r \to \infty$ (see Seneta [**6**, p. 21]). Of course, a real-valued function on $\mathbf{R}^k$ cannot possess an inverse, but we do have the following result which serves as an analogue to the asymptotic inverse.

LEMMA 3.3. *If F varies regularly and* (3.1) *holds with $\beta = \mathrm{index}(R) > 0$, then there exists $g\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ regularly varying and continuous such that*

$$\lim_{r\to\infty} F(g(r)^{-1}x_r)/r = \varphi(x) \tag{3.5}$$

whenever $x_r \to x$ in Γ.

PROOF. Let h be an asymptotic inverse of R. It follows easily that (3.5) holds with $g = f \circ h$. Since h varies regularly with $\mathrm{index}(1/\beta)$, an application of uniform convergence (Theorem 2.2) shows that g varies regularly with $\mathrm{index}(\beta^{-1}E)$. Now this g is not necessarily continuous, but suppose we define

$$\hat{g}(r) = (1-\lambda)g(n) + \lambda g(n+1), \tag{3.6}$$

where $t \in [n, n+1)$ and $\lambda = t - n$. By another application of uniform convergence, we see that $\hat{g} \sim g$ (i.e. $\hat{g}(r)g(r)^{-1} \to I$ as $r \to \infty$) and so $\hat{g}$ varies regularly and (3.5) remains true with $\hat{g}$ in place of g.

We conclude this section with two results describing the asymptotic behavior of a regularly varying function on $\mathbf{R}^k$.

THEOREM 3.4. *If F varies regularly at infinity and $E = \alpha I$ ($\alpha \neq 0$), then $F(rx)$ is a regularly varying function of $r > 0$ with index $\rho = (-\beta/\alpha)$. Furthermore we have*

$$F(r\lambda x)/F(rx) \to \lambda^\rho \quad \text{as } r \to \infty \tag{3.7}$$

uniformly on compact subsets of $x \in \Gamma$ for all $\lambda > 0$.

PROOF. We have $\varphi(\lambda x)/\varphi(x) = \lambda^\rho$ for all $\lambda > 0$ and all $x \in \Gamma$. Suppose $x_r \to x$ in Γ. For each $r > 0$ sufficiently large let

$$t(r) = \sup\{t\colon \|f(t)rx_r\| \geq 1\} \tag{3.8}$$

and note that $t = t(r)$ tends to infinity as $r \to \infty$. Let $y_t = f(t)rx_r$. If for some $r_k \to \infty$ we have $y_{t_k} \to y$ in Γ, then we have

$$\frac{F(r_k\lambda x_{r_k})}{F(r_k x_{r_k})} = \frac{F(f(t_k)^{-1}\lambda y_{t_k})}{F(f(t_k)^{-1}y_{t_k})} \to \frac{\varphi(\lambda y)}{\varphi(y)} \tag{3.9}$$

by regular variation of F, and we have already noted that the term on the right is λ^ρ. In order to complete the proof now, it suffices to show that $\{y_t\colon t \geq t_0\}$ is relatively compact for some t_0. But this follows easily from (3.8) using Theorem 2.3.

In all known cases of practical interest, the limit φ in the definition (3.1) is bounded away from 0 and ∞ on compact subsets of Γ. Under this relatively unrestrictive assumption on φ, we can derive a sweeping generalization of the preceding theorem. We state our result in terms of a generalization of the concept of regular variation known as R-0 variation. Roughly speaking, a Borel measurable function $R\colon (0,\infty) \to (0,\infty)$ is R-0 varying at infinity if as $r \to \infty$ the function $R(r)$ grows no faster than some power of r, and no slower than some other power of r. To be precise, R is R-0 varying at infinity if there exist positive constants $A > 0$, $a > 1$, $m < 1$, $M > 1$ such that

$$m \le R(\lambda r)/R(r) \le M \tag{3.10}$$

whenever $1 \le \lambda \le a$ and $r \ge A$. The appendix of Seneta's monograph [**6**] is a convenient reference on R-0 variation.

THEOREM 3.5. *Suppose F varies regularly at infinity and φ is bounded away from 0 and ∞ on compact subsets of Γ. Then $F(rx)$ is an R-0 varying function of $r > 0$ for all $x \in \Gamma$. Furthermore, if K is a compact subset of Γ and $\varepsilon > 0$, then there exists $r_0 > 0$ and $\lambda_0 > 1$ such that*

$$\lambda_0^{m-\varepsilon} \le F(\lambda r x)/F(rx) \le \lambda_0^{M+\varepsilon} \tag{3.11}$$

whenever $r \ge r_0$, $1 \le \lambda \le \lambda_0$, and $x \in K$. If $\lambda_1, \ldots, \lambda_j$ are the eigenvalues of E in (3.2), then

$$\begin{aligned} m &= \min\{0, -\beta/\operatorname{Re}(\lambda_1), \ldots, -\beta/\operatorname{Re}(\lambda_j)\}, \\ M &= \max\{0, -\beta/\operatorname{Re}(\lambda_1), \ldots, -\beta/\operatorname{Re}(\lambda_j)\}. \end{aligned} \tag{3.12}$$

PROOF. Given $x_r \to x$ in Γ, define $t(r)$ and y_t as in the proof of the preceding theorem. As before, we have that $\{y_t\colon t \ge t_0\}$ is relatively compact for some t_0, and once again (3.9) holds on convergent subsequences in $\{y_t\}$. It remains to show that we can bound $\varphi(\lambda x)/\varphi(x)$. For the case $\beta = 0$ the result is obvious. Otherwise we need to make a straightforward computation, which we illustrate in the case $\beta > 0$ for the right-hand inequality in (3.11). Let $A = \beta^{-1}E$ so that $\lambda\varphi(x) = \varphi(\lambda^A x)$ in general. All eigenvalues of A have positive real part equal to or exceeding $c = 1/M$. Without loss of generality $K = \{x\colon a \le \|x\| \le b\}$. For all $0 < \alpha < c$ for some $\lambda_0 > 1$, we have $\|\lambda^A x\| > \lambda^\alpha$ for all $\lambda \ge \lambda_0$ and all $x \in K$. Let $\mu_0 = \lambda_0^\alpha/b$. Given $x \in K$ and $1 \le \mu \le \mu_0$ we can choose λ positive and $x' \in K$ such that $\mu x = \lambda^A x'$. Since $\|\lambda^A x'\| = \|\mu x\| \le \mu_0 b = \lambda_0^\alpha$, we must have $\lambda \le \lambda_0$. But then

$$\frac{\varphi(\mu x)}{\varphi(x)} = \frac{\lambda\varphi(x')}{\varphi(x)} \le B(b\mu_0)^{1/\alpha}, \tag{3.13}$$

where B is an upper bound of $\varphi(x')/\varphi(x)$ for x, x' in K. If λ_0 is sufficiently large, then the right-hand expression in (3.11) is larger than that in (3.13) for α sufficiently close to c.

4. Regularly varying measures on $\mathbf{R}^k$. The contents of this section have no analogue in the one variable theory of regular variation for the simple reason that the asymptotic behavior of a σ-finite Borel measure on $\mathbf{R}^+$ can be completely described in terms of its distribution function. However, as we pointed out in [**4**], the definition we present below can be useful even in $\mathbf{R}^1$.

Let M denote the class of σ-finite Borel measures on $\Gamma = \mathbf{R}^k - \{0\}$ with the topology of weak convergence. In this topology we have $\nu_n \to \nu$ in M if and only if $\nu_n(A) \to \nu(A)$ for all compact Borel subsets of Γ for which $\nu(\partial A) = 0$. We will say that $\mu \in M$ varies regularly if there exist $f\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ and $R\colon \mathbf{R}^+ \to \mathbf{R}^+$, both regularly varying, such that as $r \to \infty$

$$\mu\{f(r)^{-1}\,dx\}/R(r) \to \varphi\{dx\} \tag{4.1}$$

for some full $\varphi \in M$. (A Borel measure on $\mathbf{R}^k$ is said to be full if it cannot be supported on any proper subspace.) It follows from (4.1) that

$$\lambda^\beta \varphi\{dx\} = \varphi\{\lambda^{-E}\,dx\} \tag{4.2}$$

for all $\lambda > 0$, where E and β are the indices of regular variation of f and R, respectively. The argument is essentially the same as for Lemma 3.1.

The following results on regularly varying measures are stated without proof since the proofs are essentially just a repetition of what was laid out in the previous section.

LEMMA 4.1. *μ varies regularly at infinity (zero) if and only if, for some Borel set B in Γ and some $f\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ regularly varying with index E, where every eigenvalue of E has negative (resp., positive) real part, as $r \to \infty$ we have*

$$\mu\{f(r)^{-1}\,dx\}/\mu(f(r)^{-1}B) \to \gamma\{dx\} \tag{4.3}$$

for some full $\gamma \in M$.

LEMMA 4.2. *If μ varies regularly and (4.1) holds with $\beta = \mathrm{index}(R) > 0$, then there exists $g\colon \mathbf{R}^+ \to \mathrm{GL}(\mathbf{R}^k)$ regularly varying and continuous such that, as $r \to \infty$,*

$$\mu\{g(r)^{-1}\,dx\}/r \to \varphi\{dx\}. \tag{4.4}$$

THEOREM 4.3. *If μ varies regularly at infinity, then*

$$R(r,x) = \mu\{y \in \mathbf{R}^k\colon |\langle x, y\rangle| > r\} \tag{4.5}$$

is an R-0 varying function of $r > 0$ for all x in Γ. Moreover, if we define m, M as in (3.12), then for any compact subset K of Γ and any $\varepsilon > 0$ there exist $r_0 > 0$ and $\lambda_0 > 1$ such that

$$\lambda_0^{m-\varepsilon} \le R(\lambda r, x)/R(r,x) \le \lambda_0^{M+\varepsilon} \tag{4.6}$$

whenever $r \ge r_0$, $1 \le \lambda \le \lambda_0$, and $x \in K$.

REFERENCES

1. L. deHaan, E. Omey, and S. Resnick, *Domains of attraction and regular variation in* $\mathbf{R}^d$, J. Multivariate Anal. **14** (1984), 17–33.
2. A. L. Jakimiv, *Many-dimensional Tauberian theorems and their application to Bellman-Harris branching processes*, Math. Sb. **115 (157)** (1981), 463–477. (Russian)
3. M. Meerschaert, *Multivariable domains of attraction and regular variation*, Doctoral Dissertation, University of Michigan, 1984.

4. ——, *Regular variation and domains of attraction in* $\mathbf{R}^k$, Statist. Probab. Lett. **4** (1986), 43–45.
5. ——, *Domains of attraction of nonnormal operator-stable laws*, J. Multivariate Anal. **19** (1986), 342–347.
6. E. Seneta, *Regularly varying functions*, Lecture Notes in Math., vol. 508, Springer-Verlag, New York and Berlin, 1976.
7. A. Stam, *Regular variation in* $\mathbf{R}^d$ *and the Abel-Tauber Theorem*, Technical Report T. W. 189, Mathematisch Instituut Rigksuniversiteit Groningen, The Netherlands.

DEPARTMENT OF MATHEMATICS, ALBION COLLEGE, ALBION, MICHIGAN 49224