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ON EXTREME POINTS OF FAMILIES DESCRIBED BY SUBORDINATION

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ABSTRACT. Let $s(F)$ denote the set of analytic functions in $D = \{z: |z| < 1\}$ subordinate to an analytic function F. It is shown that if F is a polynomial then the extreme points of the closed convex hull of $s(F) \subset \{F \circ \phi: \phi \in$ extreme points of $B(H_0^\infty)\}$. Also if $F(z) = ((z-\alpha)/(1-\bar{\alpha}z))^n$, $|\alpha| < 1$ and n is a positive integer then the extreme points of the closed convex hull of $s(F) = \{F \circ \phi: \phi \in$ extreme points of $B(H_0^\infty)\}$. An analogue of Ryff's theorem, and other results related to subordination in Bergman spaces have been obtained.

1. Introduction. Let D denote the open unit disc in the complex plane, and let A denote the set of functions analytic in D. Then A is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of D. A function $f \in A$ is said to be in the space H^p $(1 \leq p < \infty)$ if

$$||f||_p = \lim_{r\to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

The Bergman space A^p $(1 \leq p < \infty)$ is the set of all functions f in A such that

$$||f||_{A^p} = \left\{ \iint\limits_D |f(z)|^p \, dA(z) \right\}^{1/p} < \infty.$$

We note that $H^p \subset A^p$.

Let $B(H^\infty)$ denote the subset of A consisting of functions φ satisfying $|\varphi(z)| < 1$ for $|z| < 1$, and $B(H_0^\infty)$ denote those functions in $B(H^\infty)$ which vanish at the origin. A function f is said to be subordinate to F if $f = F \circ \phi$ for some $\phi \in B(H_0^\infty)$. This relation will be denoted by $f \prec F$. We let $s(F)$ denote the set $\{f: f \prec F\}$, and for any compact subset H of A, we let $s(H) = \{f: f \prec F$ for some $F \in H\}$. The closed convex hull of $s(F)$ will be denoted by $Hs(F)$, and let $\operatorname{Ext} Hs(F)$ denote the set of all extreme points of $Hs(F)$. In case F is univalent, then $f \prec F$ is equivalent to saying that $f(0) = F(0)$ and $f(D) \subset F(D)$.

Let F be a function in A. Then in [**4**] it is shown that $\{F(xz): |x| = 1\} \subset \operatorname{Ext} Hs(F)$. In case F is univalent and $F(D)$ is a Jordan domain then $\operatorname{Ext} Hs(F) \subset \{F \circ \phi: \phi \in \operatorname{Ext} B(H_0^\infty)\}$ [**1**; **5**, p. 143]. Recently, it has been announced by Y. Abu-Muhanna and D. Hallenbeck (Abstracts Amer. Math. Soc. **7** (1986), p. 254)

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that the above result is true for any univalent function F. Another result in [**4**] says that if $F \in H^p$ for some p $(1 < p < \infty)$ and $f \in Hs(F)$ such that $||f||_p = ||F||_p$ then $f \in \operatorname{Ext} Hs(F)$. For more details, see Chapter 8 in [**5**]. In this paper, we will use the fact [**6**] that if $f \in \operatorname{Ext} Hs(F)$, then $f = F \circ \phi$ for some $\phi \in B(H_0^\infty)$. The characterization of $\operatorname{Ext} B(H^\infty)$ can be found in [**7**, p. 138].

The object of this paper is to study the set $\operatorname{Ext} Hs(F)$ when F is a nonunivalent function in A. The most natural candidate for such functions are the polynomials. The first result in this direction is Theorem 1 of §2, which states that if F is a polynomial then $\operatorname{Ext} Hs(F) \subset \{F \circ \phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}$. Another result is Theorem 2 of the same section, which says that if $F_\alpha(z) = ((z-\alpha)/(1-\bar{\alpha}z))^n$, $|\alpha| < 1$ and n is a positive integer, then $\operatorname{Ext} Hs(F_\alpha) = \{F_\alpha \circ \phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}$.

In §3, an analogue of Ryff's theorem [**10**, Theorem 3] is obtained for Bergman spaces (Theorem 3). Other results related to A^p subordination are discussed.

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2. Families described by subordination. Recently, it has been announced (see the introduction) that if F is a univalent function then $\operatorname{Ext} Hs(F) \subset \{F\circ\phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}$.

In the next theorem, we consider an arbitrary polynomial, which need not be univalent.

THEOREM 1. *If F is a polynomial then*

$$\operatorname{Ext} Hs(F) \subset \{F \circ \phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}.$$

First, let us consider a special case illustrating the idea of the proof before presenting the complete form.

Let $F(z) = z + z^2$, and suppose that $F \circ \phi$ is an extreme point of $Hs(F)$, $\phi \in B(H_0^\infty)$. If ϕ were not an extreme point of $B(H_0^\infty)$, then by [**7**, p. 139], there exists $h \in B(H^\infty)$, $h \neq 0$, such that $|h| = 1 - |\phi|$ a.e. on ∂D. Thus $|\phi + azh| \leq 1$ for $|a| \leq 1$. Let 1 and w be the square roots of unity, and $\alpha^2 = -1$. Then it is easy to establish the following identities:

(1) $\phi = \frac{1}{2}(\phi + zh) + \frac{1}{2}(\phi + wzh)$.

(2) $\phi^2 + z^2h^2 = \frac{1}{2}(\phi + zh)^2 + \frac{1}{2}(\phi + wzh)^2$.

(3) $\phi^2 - z^2h^2 = \frac{1}{2}(\phi + \alpha zh)^2 + \frac{1}{2}(\phi + \alpha wzh)^2$.

(4) $\phi = \frac{1}{2}(\phi + \alpha zh) + \frac{1}{2}(\phi + \alpha wzh)$.

From (1) and (2) we get

(5) $\phi + (\phi^2 + z^2h^2) = \frac{1}{2}[(\phi + zh) + (\phi + zh)^2] + \frac{1}{2}[(\phi + wzh) + (\phi + wzh)^2]$.

From (3) and (4) we get

(6) $\phi + (\phi^2 - z^2h^2) = \frac{1}{2}[(\phi + \alpha zh) + (\phi + \alpha zh)^2] + \frac{1}{2}[(\phi + \alpha wzh) + (\phi + \alpha wzh)^2]$.

Since $|\phi + zh| \leq 1$, $|\phi + wzh| \leq 1$, $|\phi + \alpha zh| \leq 1$ and $|\phi + \alpha wzh| \leq 1$, and each one vanishes at the origin, we get from (5) and (6)

(7) $h_1 = \phi + (\phi^2 + z^2h^2)$ and $h_2 = \phi + (\phi^2 - z^2h^2)$ belong to $Hs(F)$.

Thus by (7), $F \circ \phi = \phi + \phi^2 = \frac{1}{2}h_1 + \frac{1}{2}h_2$. Since $h \neq 0$ we get $h_1 \neq h_2$. Consequently $F \circ \phi$ cannot be an extreme point of $Hs(F)$. This contradiction proves Theorem 1 in case $F(z) = z + z^2$.

PROOF OF THEOREM 1. Let $F(z) = A_0 + A_1z + A_2z^2 + \cdots + A_{n-1}z^{n-1} + A_nz^n$, $A_n \neq 0$, and let $F \circ \phi$ be an extreme point of $Hs(F)$, $\phi \in B(H_0^\infty)$. If ϕ were not an extreme point of $B(H_0^\infty)$, then the argument used in the illustration of Theorem 1 shows that there exists $h \in B(H^\infty)$, $h \neq 0$, such that for any θ, $0 \leq \theta < 2\pi$, we have $|\phi \pm e^{i\theta} zh| \leq 1$.

Let $1, w, w^2, \ldots, w^{n-1}$ be the nth roots of unity, and $\alpha^n = -1$. Then we claim the following:

(1) For $k = 1, 2, \ldots, n-1$, $n\phi^k = (\phi + wzh)^k + \cdots + (\phi + w^{n-1}zh)^k + (\phi + zh)^k$.

(2) $n(\phi^n + z^n h^n) = (\phi + wzh)^n + \cdots + (\phi + w^{n-1}zh)^n + (\phi + zh)^n$.

(3) $n(\phi^n - z^n h^n) = (\phi + \alpha wzh)^n + \cdots + (\phi + \alpha w^{n-1}zh)^n + (\phi + \alpha zh)^n$.

(4) For $k = 1, 2, \ldots, n-1$, $n\phi^k = (\phi + \alpha wzh)^k + \cdots + (\phi + \alpha w^{n-1}zh)^k + (\phi + \alpha zh)^k$.

To prove (1), we write

$$(\phi + wzh)^k = \phi^k + \binom{k}{1}\phi^{k-1}wzh + \binom{k}{2}\phi^{k-2}w^2z^2h^2 + \cdots + w^k z^k h^k,$$

$$(\phi + w^2zh)^k = \phi^k + \binom{k}{1}\phi^{k-1}w^2zh + \binom{k}{2}\phi^{k-2}(w^2)^2z^2h^2 + \cdots + (w^k)^2 w^k h^k,$$

$$\vdots$$

$$(\phi + w^{n-1}zh)^k = \phi^k + \binom{k}{1}\phi^{k-1}w^{n-1}zh + \binom{k}{2}\phi^{k-2}(w^2)^{n-1}z^2h^2 + \cdots + (w^k)^{n-1}z^k h^k,$$

$$(\phi + zh)^k = \phi^k + \binom{k}{1}\phi^{k-1}zh + \binom{k}{2}\phi^{k-2}z^2h^2 + \cdots + z^k h^k.$$

By adding these identities and taking into account the fact that $1 + (w^k) + (w^k)^2 + \cdots + (w^k)^{n-1} = 0$ for $k = 1, 2, 3, \ldots, n-1$, we get (1).

To prove (2), we write

$$(\phi + wzh)^n = \phi^n + \binom{n}{1}\phi^{n-1}wzh + \binom{n}{2}\phi^{n-2}w^2z^2h^2 + \cdots + w^n z^n h^n,$$

$$(\phi + w^2zh)^n = \phi^n + \binom{n}{1}\phi^{n-1}w^2zh + \binom{n}{2}\phi^{n-2}(w^2)^2z^2h^2 + \cdots + (w^n)^2 z^n h^n,$$

$$\vdots$$

$$(\phi + w^{n-1}zh)^n = \phi^n + \binom{n}{1}\phi^{n-1}w^{n-1}zh + \binom{n}{2}\phi^{n-2}(w^2)^{n-1}z^2h^2 + \cdots + (w^n)^{n-1}z^n h^n,$$

$$(\phi + zh)^n = \phi^n + \binom{n}{1}\phi^{n-1}zh + \binom{n}{2}\phi^{n-2}z^2h^2 + \cdots + z^n h^n.$$

By adding these identities and taking into account the fact that $w^n = 1$ and $1 + (w^k) + (w^k)^2 + \cdots + (w^k)^{n-1} = 0$ for $k = 1, 2, \ldots, n-1$, we get identity (2).

Similarly, we obtain (3) and (4). Now, from (1) and (2) we have:
(5)

$$\begin{aligned}
h_1 &= A_0 + A_1\phi + A_2\phi^2 + \cdots + A_n\phi^n + A_n z^n h^n \\
&= \frac{1}{n}[A_0 + A_1(\phi + wzh) + A_2(\phi + wzh)^2 + \cdots + A_n(\phi + wzh)^n] \\
&\quad + \frac{1}{n}[A_0 + A_1(\phi + w^2zh) + A_2(\phi + w^2zh)^2 + \cdots + A_n(\phi + w^2zh)^n] \\
&\quad \vdots \\
&\quad + \frac{1}{n}[A_0 + A_1(\phi + w^{n-1}zh) + A_2(\phi + w^{n-1}zh)^2 + \cdots + A_n(\phi + w^{n-1}zh)^n] \\
&\quad + \frac{1}{n}[A_0 + A_1(\phi + zh) + A_2(\phi + zh)^2 + \cdots + A_n(\phi + zh)^n] \\
&= \frac{1}{n}[F(\phi + wzh) + F(\phi + w^2zh) + \cdots + F(\phi + zh)].
\end{aligned}$$

Thus $h_1 \in Hs(F)$. Similarly, from (3) and (4) we obtain $h_2 = A_0 + A_1\phi + A_2\phi^2 + \cdots + A_n\phi^n - A_n z^n h^n \in Hs(F)$. Finally, since $F \circ \phi = \frac{1}{2}h_1 + \frac{1}{2}h_2$, and $h \neq 0$, $F \circ \phi$ cannot be an extreme point of $Hs(F)$. This contradiction completes the proof of Theorem 1.

If F is an analytic function in D, then by [**4**], $\{F(xz)\colon |x|\} \subset \operatorname{Ext} Hs(F)$. If c is a complex number satisfying $|c| \leq 1$ and $c \neq -1$ and $F_\alpha(z) = ((1 + cz)/(1 - z))^\alpha$, then it is known [**2**] that $\operatorname{Ext} Hs(F_\alpha) = \{F_\alpha(xz)\colon |x| = 1\}$. The following theorem is suggested by the above results and Theorem 1.

THEOREM 2. *Let α be a complex number satisfying $|\alpha| < 1$, and let n be a positive integer. If $F_\alpha(z) = ((z - \alpha)/(1 - \bar{\alpha}z))^n$, then*

$$\operatorname{Ext} Hs(F_\alpha) = \{F_\alpha \circ \phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}.$$

PROOF. Let $((\phi - \alpha)/(1 - \bar{\alpha}\phi))^n$ be an extreme point of $Hs(F_\alpha)$, $\phi \in B(H_0^\infty)$. Set $b = (\phi - \alpha)/(1 - \bar{\alpha}\phi)$. If ϕ were not an extreme point of $B(H_0^\infty)$, then by [**9**, Lemma 7], b is not an extreme point of $B(H^\infty)$. As in the illustration of Theorem 1, there exists $h \in B(H^\infty)$, $h \neq 0$, such that $|h| = 1 - |b|$ a.e. on ∂D. Consequently, $|b \pm e^{i\theta}zh| \leq 1$ for $0 \leq \theta < 2\pi$. Since $(b \pm e^{i\theta}zh)(0) = -\alpha$ and $(z - \alpha)/(1 - \bar{\alpha}z)$ is univalent, we get $(b \pm e^{i\theta}zh) \in s((z - \alpha)/(1 - \bar{\alpha}z))$. Thus $(b \pm e^{i\theta}zh)^n \in s(F_\alpha) \cdots (1)$.

Let $1, w, w^2, \ldots, w^{n-1}$ be the nth roots of unity. Then as in the proof of Theorem 1, we have:

(2) $n(b^n + z^n h^n) = (b + wzh)^n + (b + w^2zh)^n + \cdots + (b + w^{n-1}zh)^n + (b + zh)^n$.

(3) $n(b^n - z^n h^n) = (b + \alpha wzh)^n + (b + \alpha w^2zh)^n + \cdots + (b + \alpha w^{n-1}zh)^n + (b + \alpha zh)^n$, where $\alpha^n = -1$.

From (1), (2) and (3) we get $b^n + z^n h^n$ and $b^n - z^n h^n \in Hs(F_\alpha)$. Finally, since $b^n = \frac{1}{2}(b^n + z^n h^n) + \frac{1}{2}(b^n - z^n h^n)$ and $h \neq 0$, then b^n cannot be an extreme point of $Hs(F_\alpha)$. This contradiction completes the proof that $\operatorname{Ext} Hs(F_\alpha) \subset \{F_\alpha \circ \phi\colon \phi \in \operatorname{Ext} B(H_0^\infty)\}$.

On the other hand, let $\phi \in \operatorname{Ext} B(H_0^\infty)$. Since F_α is a finite Blaschke product, by [**9**, Theorem 9] we have $F_\alpha \circ \phi$ is an extreme point of $B(H^\infty)$. Since $Hs(F_\alpha) \subset B(H^\infty)$, we conclude that $F_\alpha \circ \phi$ is an extreme point of $Hs(F_\alpha)$. This completes the proof of Theorem 2.

3. An analogue of Ryff's theorem in Bergman spaces. In [**10**], Ryff proved the following remarkable result: Let $F \in H^p$, $0 < p < \infty$, and $f \prec F$. Then $\|f\|_p = \|F\|_p$ if and only if $f = F \circ \phi$ for some inner function ϕ with $\phi(0) = 0$.

Concerning A^p spaces, Ryff's theorem is not true in general. For example, let $F(z) = z$ and $f(z) = z^2$. Then $f \prec F$ and it is easy to see that $\|f\|_{A^p} \neq \|F\|_{A^p}$ for any p, $0 < p < \infty$. This suggests the following analogue of Ryff's theorem.

THEOREM 3. *Let F be a function in A^p, $0 < p < \infty$, and $f \prec F$. Then $\|f\|_{A^p} = \|F\|_{A^p}$ if and only if $f = F \circ \phi$, $\phi(z) = cz$ for some c, $|c| = 1$.*

PROOF. Let $\|f\|_{A^p} = \|F\|_{A^p}$. Thus

$$(1) \qquad \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, r dr = \int_0^1 \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \, r dr.$$

For $g \in A$ and $0 < p < \infty$, let $m_p(g, r) = ((1/2\pi) \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta)^{1/p}$. For $0 < r < 1$, let $h(r) = m_p^p(F, r) - m_p^p(f, r)$. By the Littlewood subordination theorem [**3**, p. 191], h is nonnegative on $(0, 1)$, and clearly it is continuous there. This together with equation (1) implies that $h \in L^1[[0, 1], rdr]$. Since $\int_0^1 h(r) \, r dr = 0$ and h is continuous on $(0, 1)$ we get $h(r) = 0$ on $(0, 1)$. Thus $m_p(f, r) = m_p(F, r)$. Hence by [**3**, p. 191], $f(z) = F(cz)$, $|c| = 1$, as required.

Conversely, let $f(z) = F(cz)$, $|c| = 1$. Put $c = \cos\alpha + i \sin\alpha$ and $z = x + iy$. Then $cz = (x\cos\alpha - y\sin\alpha) + i(y\cos\alpha + x\sin\alpha)$. Let $T\colon (x, y) \to (u, v)$, where u and v are the real and imaginary parts of cz. Then the Jacobian of T (denoted by J_T) is c^2. By using change of variables, we get $\int_D \int |F(x, y)|^p \, dx \, dy = \int_D \int |F(u, v)|^p |J_T| \, du \, dv$. Thus $\|F\|_{A^p} = \|f\|_{A^p}$. This completes the proof of Theorem 3.

In [**4**] (see [**5**, p. 124]) the following result is proved.

THEOREM A. *Let $F \in H^p$ for some p, $1 < p < \infty$, and $f \in Hs(F)$. If $\|f\|_p = \|F\|_p$ then $f \in \operatorname{Ext} Hs(F)$.*

The next result is the analogue of Theorem A in A^p spaces.

THEOREM 4. *Let $F \in A^p$, $1 < p < \infty$, and $f \in Hs(F)$. If $\|f\|_{A^p} = \|F\|_{A^p}$ then $f \in \operatorname{Ext} Hs(F)$.*

PROOF. First, we show that if $F \in A^p$ $(p \geq 1)$ then $Hs(F) \subset A^p$, and moreover if $f \in Hs(F)$ then $\|f\|_{A^p} \leq \|F\|_{A^p}$. One way to see this is to use an argument similar to the one used in [**4**, Theorem 7]. So let $f \in Hs(F)$. Then f can be approximated uniformly on compact subsets of D, by functions of the form $\sum_{k=1}^n \lambda_k f_k$, where $\lambda_k \geq 0$, $\sum_{k=1}^n \lambda_k = 1$ and $f_k \in s(F)$. By the Littlewood subordination theorem we have $m_p(f_k, r) \leq m_p(F, r)$ for $0 < r < 1$ and $k = 1, 2, \ldots, n$. By the Minkowski inequality we get

$$m_p\left(\sum_{k=1}^n \lambda_k f_k, r\right) \leq \sum_{k=1}^n \lambda_k m_p(f_k, r) \leq m_p(F, r).$$

Consequently, $m_p(f, r) \leq m_p(F, r)$ for $0 < r < 1$. Thus we get

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, r dr \leq \int_0^1 \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \, r dr = \|F\|_{A^p}^p.$$

Thus $\|f\|_{A^p} \leq \|F\|_{A^p}$, as required.

Now, suppose that $f = \lambda g + (1-\lambda)h$, $g, h \in Hs(F)$. Let $f_1 = f/||F||_{A^p}$, $g_1 = g/||F||_{A^p}$ and $h_1 = h/||F||_{A^p}$. From above, $||g_1||_{A^p} \le 1$ and $||h_1||_{A^p} \le 1$. Since $f_1 = \lambda g_1 + (1-\lambda)h_1$, $||f_1||_{A^p} = 1$ and A^p is a strictly convex space, we get $g_1 = h_1 = f_1$. Consequently, $g = h = f$. This proves that $f \in \operatorname{Ext} Hs(F)$, and that ends the proof of the theorem.

In case $F \in H^p$, $1 \le p < \infty$, we have the following

COROLLARY 5. *Let $F \in H^p$, $1 \le p < \infty$, and $f \in Hs(F)$. If $||f||_{A^r} = ||F||_{A^r}$ for some r, $1 < r \le 2p$, then $f \in s(F)$.*

The proof of Corollary 5 requires the following known result.

THEOREM B [**8**, **11**]. $H^p \subset A^{2p}$, $0 < p < \infty$.

PROOF OF COROLLARY 5. Since $F \in H^p$, then by the proof of Theorem 7 in [**4**] we have $Hs(F) \subset H^p$. Thus by Theorem B, $Hs(F) \subset A^{2p}$. If $1 < r \le 2p$, then $Hs(F) \subset A^r$. If $f \in Hs(F)$ with $||f||_{A^r} = ||F||_{A^r}$, then by Theorem 4, f is an extreme point of $Hs(F)$. Consequently, by [**6**] $f \in s(F)$ and this completes the proof of Corollary 5.

ADDED IN PROOF. Recently the author obtained the following: If P is a polynomial and F is univalent then $\operatorname{Ext} Hs(P \circ F) \subseteq \{(P \circ F) \circ \phi \colon \phi \in B(H_0^B)\}$.

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