ON EXTREME POINTS OF FAMILIES DESCRIBED BY SUBORDINATION

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ABSTRACT. Let $s(F)$ denote the set of analytic functions in $D = \{z: |z| < 1\}$ subordinate to an analytic function $F$. It is shown that if $F$ is a polynomial then the extreme points of the closed convex hull of $s(F) \subset \{F \circ \phi: \phi \in \text{extreme points of } B(H_0^\infty)\}$. Also if $F(z) = ((z - \alpha)/(1 - \bar{\alpha}z))^n$, $|\alpha| < 1$ and $n$ is a positive integer then the extreme points of the closed convex hull of $s(F) = \{F \circ \phi: \phi \in \text{extreme points of } B(H_0^\infty)\}$. An analogue of Ryff's theorem, and other results related to subordination in Bergman spaces have been obtained.

1. Introduction. Let $D$ denote the open unit disc in the complex plane, and let $A$ denote the set of functions analytic in $D$. Then $A$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $D$. A function $f \in A$ is said to be in the space $H^p$ ($1 \leq p < \infty$) if

$$
||f||_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.
$$

The Bergman space $A^p$ ($1 \leq p < \infty$) is the set of all functions $f$ in $A$ such that

$$
||f||_{A^p} = \left\{ \int_D |f(z)|^p \, dA(z) \right\}^{1/p} < \infty.
$$

We note that $H^p \subset A^p$.

Let $B(H^\infty)$ denote the subset of $A$ consisting of functions $\varphi$ satisfying $|\varphi(z)| < 1$ for $|z| < 1$, and $B(H_0^\infty)$ denote those functions in $B(H^\infty)$ which vanish at the origin. A function $f$ is said to be subordinate to $F$ if $f = F \circ \phi$ for some $\phi \in B(H_0^\infty)$. This relation will be denoted by $f < F$. We let $s(F)$ denote the set $\{f: f < F\}$, and for any compact subset $H$ of $A$, we let $s(H) = \{f: f < F$ for some $F \in H\}$. The closed convex hull of $s(F)$ will be denoted by $Hs(F)$, and let $\text{Ext } Hs(F)$ denote the set of all extreme points of $Hs(F)$. In case $F$ is univalent, then $f < F$ is equivalent to saying that $f(0) = F(0)$ and $f(D) \subset F(D)$.

Let $F$ be a function in $A$. Then in [4] it is shown that $\{F(xz): |x| = 1\} \subset \text{Ext } Hs(F)$. In case $F$ is univalent and $F(D)$ is a Jordan domain then $\text{Ext } Hs(F) \subset \{F \circ \phi: \phi \in \text{Ext } B(H_0^\infty)\}$ [1; 5, p. 143]. Recently, it has been announced by Y. Abu-Muhanna and D. Hallenbeck (Abstracts Amer. Math. Soc. 7 (1986), p. 254)

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that the above result is true for any univalent function $F$. Another result in [4] says that if $F \in H^p$ for some $p (1 < p < \infty)$ and $f \in Hs(F)$ such that $\|f\|_p = \|F\|_p$ then $f \in \text{Ext} Hs(F)$. For more details, see Chapter 8 in [5]. In this paper, we will use the fact [6] that if $f \in \text{Ext} Hs(F)$, then $f = F \circ \phi$ for some $\phi \in B(H^\infty_0)$. The characterization of $\text{Ext} B(H^\infty_0)$ can be found in [7, p. 138].

The object of this paper is to study the set $\text{Ext} Hs(F)$ when $F$ is a nonunivalent function in $A$. The most natural candidate for such functions are the polynomials. The first result in this direction is Theorem 1 of §2, which states that if $F$ is a polynomial then $\text{Ext} Hs(F) \subset \{F \circ \phi : \phi \in \text{Ext} B(H^\infty_0)\}$. Another result is Theorem 2 of the same section, which says that if $F_\alpha(z) = ((z - \alpha)/(1 - \alpha z))^n$, $|\alpha| < 1$ and $n$ is a positive integer, then $\text{Ext} Hs(F_\alpha) = \{F_\alpha \circ \phi : \phi \in \text{Ext} B(H^\infty_0)\}$.

In §3, an analogue of Ryff’s theorem [10, Theorem 3] is obtained for Bergman spaces (Theorem 3). Other results related to $A^p$ subordination are discussed.

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2. Families described by subordination. Recently, it has been announced (see the introduction) that if $F$ is a univalent function then $\text{Ext} Hs(F) \subset \{F \circ \phi : \phi \in \text{Ext} B(H^\infty_0)\}$.

In the next theorem, we consider an arbitrary polynomial, which need not be univalent.

**THEOREM 1.** If $F$ is a polynomial then

$$\text{Ext} Hs(F) \subset \{F \circ \phi : \phi \in \text{Ext} B(H^\infty_0)\}.$$  

First, let us consider a special case illustrating the idea of the proof before presenting the complete form.

Let $F(z) = z + z^2$, and suppose that $F \circ \phi$ is an extreme point of $Hs(F)$, $\phi \in B(H^\infty_0)$. If $\phi$ were not an extreme point of $B(H^\infty_0)$, then by [7, p. 139], there exists $h \in B(H^\infty)$, $h \neq 0$, such that $|h| = 1 - |\phi|$ a.e. on $\partial D$. Thus $|\phi + \alpha h| \leq 1$ for $|\alpha| \leq 1$. Let $1$ and $w$ be the square roots of unity, and $\alpha^2 = -1$. Then it is easy to establish the following identities:

1. $\phi = \frac{1}{2}(\phi + zh) + \frac{1}{2}(\phi + wz)$.
2. $\phi^2 + z^2 h^2 = \frac{1}{2}(\phi + zh)^2 + \frac{1}{2}(\phi + wz)^2$.
3. $\phi^2 - z^2 h^2 = \frac{1}{2}(\phi + \alpha zh)^2 + \frac{1}{2}(\phi + \alpha wz h^2)^2$.
4. $\phi = \frac{1}{2}(\phi + \alpha zh) + \frac{1}{2}(\phi + \alpha wz)h$.

From (1) and (2) we get

$$\phi + (\phi^2 + z^2 h^2) = \frac{1}{2}[(\phi + zh) + (\phi + zh)^2] + \frac{1}{2}[(\phi + wz) + (\phi + wz)^2].$$

From (3) and (4) we get

$$\phi + (\phi^2 - z^2 h^2) = \frac{1}{2}[(\phi + \alpha zh) + (\phi + \alpha zh)^2] + \frac{1}{2}[(\phi + \alpha wz h^2) + (\phi + \alpha wz h^2)^2].$$

Since $|\phi + zh| \leq 1$, $|\phi + wz| \leq 1$, $|\phi + \alpha zh| \leq 1$ and $|\phi + \alpha wz| \leq 1$, and each one vanishes at the origin, we get from (5) and (6)

$$h_1 = \phi + (\phi^2 + z^2 h^2) \quad \text{and} \quad h_2 = \phi + (\phi^2 - z^2 h^2) \quad \text{belong to} \quad Hs(F).$$

Thus by (7), $F \circ \phi = \phi + \phi^2 = \frac{1}{2}h_1 + \frac{1}{2}h_2$. Since $h \neq 0$ we get $h_1 \neq h_2$. Consequently $F \circ \phi$ cannot be an extreme point of $Hs(F)$. This contradiction proves Theorem 1 in case $F(z) = z + z^2$.

**PROOF OF THEOREM 1.** Let $F(z) = A_0 + A_1 z + A_2 z^2 + \cdots + A_{n-1} z^{n-1} + A_n z^n$, $A_n \neq 0$, and let $F \circ \phi$ be an extreme point of $Hs(F)$, $\phi \in B(H^\infty_0)$. If $\phi$ were not an extreme point of $B(H^\infty_0)$, then the argument used in the illustration of Theorem 1
shows that there exists \( h \in B(H^\infty), h \neq 0 \), such that for any \( \theta, 0 \leq \theta < 2\pi \), we have \( |\phi \pm e^{i\theta}zh| \leq 1 \).

Let \( 1, w, w^2, \ldots, w^{n-1} \) be the \( n \)th roots of unity, and \( \alpha^n = -1 \). Then we claim the following:

1. For \( k = 1, 2, \ldots, n-1 \), \( n\phi^k = \phi + wzh) + \cdots + (\phi + w^{n-1}zh)^k + (\phi + zh)^k \).
2. \( n(\phi^n + z^n) = (\phi + wzh) + \cdots + (\phi + w^{n-1}zh)^n + (\phi + zh)^n \).
3. \( n(\phi^n - z^n) = (\phi + \alpha wzh) + \cdots + (\phi + \alpha w^{n-1}zh)^n + (\phi + \alpha zh)^n \).
4. For \( k = 1, 2, \ldots, n-1 \), \( n\phi^k = (\phi + \alpha wzh)^k + \cdots + (\phi + \alpha w^{n-1}zh)^k + (\phi + \alpha zh)^k \).

To prove (1), we write

\[
(\phi + wzh)^k = \phi^k + \binom{k}{1} \phi^{k-1}wzh + \binom{k}{2} \phi^{k-2}w^2z^2h^2 + \cdots + w^kz^kh^k,
\]

\[
(\phi + w^2zh)^k = \phi^k + \binom{k}{1} \phi^{k-1}w^2zh + \binom{k}{2} \phi^{k-2}(w^2)^2z^2h^2 + \cdots + (w^k)^2w^kh^k,
\]

\[
(\phi + w^{n-1}zh)^k = \phi^k + \binom{k}{1} \phi^{k-1}w^{n-1}zh
\]

\[
+ \binom{k}{2} \phi^{k-2}(w^{n-1}z)^2h^2 + \cdots + (w^k)^{n-1}z^kh^k,
\]

\[
(\phi + zh)^k = \phi^k + \binom{k}{1} \phi^{k-1}zh + \binom{k}{2} \phi^{k-2}z^2h^2 + \cdots + z^kh^k.
\]

By adding these identities and taking into account the fact that \( 1 + (w^k) + (w^k)^2 + \cdots + (w^k)^{n-1} = 0 \) for \( k = 1, 2, 3, \ldots, n-1 \), we get (1).

To prove (2), we write

\[
(\phi + wzh)^n = \phi^n + \binom{n}{1} \phi^{n-1}wzh + \binom{n}{2} \phi^{n-2}w^2z^2h^2 + \cdots + w^nz^nh^n,
\]

\[
(\phi + w^2zh)^n = \phi^n + \binom{n}{1} \phi^{n-1}w^2zh + \binom{n}{2} \phi^{n-2}(w^2)^2z^2h^2 + \cdots + (w^2)^2w^nh^n,
\]

\[
(\phi + w^{n-1}zh)^n = \phi^n + \binom{n}{1} \phi^{n-1}w^{n-1}zh
\]

\[
+ \binom{n}{2} \phi^{n-2}(w^{n-1}z)^2h^2 + \cdots + (w^k)^{n-1}z^nh^n,
\]

\[
(\phi + zh)^n = \phi^n + \binom{n}{1} \phi^{n-1}zh + \binom{n}{2} \phi^{n-2}z^2h^2 + \cdots + z^nh^n.
\]

By adding these identities and taking into account the fact that \( w^n = 1 \) and \( 1 + (w^k) + (w^k)^2 + \cdots + (w^k)^{n-1} = 0 \) for \( k = 1, 2, \ldots, n-1 \), we get identity (2).
Similarly, we obtain (3) and (4). Now, from (1) and (2) we have:

\[
\begin{align*}
    h_1 &= A_0 + A_1 \phi + A_2 \phi^2 + \cdots + A_n \phi^n + A_n z^n h^n \\
    &= \frac{1}{n} \left[ A_0 + A_1 (\phi + wzh) + A_2 (\phi + wzh)^2 + \cdots + A_n (\phi + wzh)^n \right] \\
    &\quad + \frac{1}{n} \left[ A_0 + A_1 (\phi + w^2 zh) + A_2 (\phi + w^2 zh)^2 + \cdots + A_n (\phi + w^2 zh)^n \right] \\
    \vdots \\
    &\quad + \frac{1}{n} \left[ A_0 + A_1 (\phi + w^{n-1} zh) + A_2 (\phi + w^{n-1} zh)^2 + \cdots + A_n (\phi + w^{n-1} zh)^n \right] \\
    &\quad + \frac{1}{n} \left[ A_0 + A_1 (\phi + zh) + A_2 (\phi + zh)^2 + \cdots + A_n (\phi + zh)^n \right] \\
    &= \frac{1}{n} \left[ F(\phi + wzh) + F(\phi + w^2 zh) + \cdots + F(\phi + zh) \right].
\end{align*}
\]

Thus \( h_1 \in Hs(F) \). Similarly, from (3) and (4) we obtain \( h_2 = A_0 + A_1 \phi + A_2 \phi^2 + \cdots + A_n \phi^n - A_n z^n h^n \in Hs(F) \). Finally, since \( F \circ \phi = \frac{1}{2} h_1 + \frac{i}{2} h_2 \), and \( h \neq 0 \), \( F \circ \phi \) cannot be an extreme point of \( Hs(F) \). This contradiction completes the proof of Theorem 1.

If \( F \) is an analytic function in \( D \), then by \([4]\), \( \{F(xz) : |x|\} \subseteq Ext Hs(F) \). If \( c \) is a complex number satisfying \( |c| < 1 \) and \( c \neq -1 \) and \( F_0(z) = ((1 + cz)/(1 - z))^\alpha \), then it is known \([2]\) that \( Ext B(H_0) = \{F_0(xz) : |x| = 1\} \). The following theorem is suggested by the above results and Theorem 1.

**THEOREM 2.** Let \( \alpha \) be a complex number satisfying \( |\alpha| < 1 \), and let \( n \) be a positive integer. If \( F_\alpha(z) = ((z - \alpha)/(1 - \bar{\alpha}z))^n \), then

\[
    Ext Hs(F_\alpha) = \{F_\alpha \circ \phi : \phi \in Ext B(H_0^\infty)\}.
\]

**PROOF.** Let \( ((\phi - \alpha)/(1 - \bar{\alpha}\phi))^n \) be an extreme point of \( Hs(F_\alpha) \), \( \phi \in B(H_0^\infty) \). Set \( b = (\phi - \alpha)/(1 - \bar{\alpha}\phi) \). If \( \phi \) were not an extreme point of \( B(H_0^\infty) \), then by \([9, \text{Lemma 7}]\), \( b \) is not an extreme point of \( B(H_0^\infty) \). As in the illustration of Theorem 1, there exists \( h \in B(H_\infty) \), \( h \neq 0 \), such that \( |h| = 1 - |b| \) a.e. on \( \partial D \). Consequently, \( |b \pm e^{i\theta}zh| \leq 1 \) for \( 0 \leq \theta < 2\pi \). Since \( (b \pm e^{i\theta}zh)(0) = -\alpha \) and \( (z - \alpha)/(1 - \bar{\alpha}z) \) is univalent, we get \( (b \pm e^{i\theta}zh) \in s((z - \alpha)/(1 - \bar{\alpha}z)) \). Thus \( (b \pm e^{i\theta}zh)^n \in s(F_\alpha) \cdots (1) \).

Let \( 1, w, w^2, \ldots, w^{n-1} \) be the \( n \)th roots of unity. Then as in the proof of Theorem 1, we have:

\[
\begin{align*}
    (2) \quad n(b^n + z^n h^n) &= (b + wzh^n) + (b + w^2 zh^n) + \cdots + (b + w^{n-1}zh^n) + (b + zh^n) \\
    (3) \quad n(b^n - z^n h^n) &= (b + \alpha wzh^n) + (b + \alpha w^2 zh^n) + \cdots + (b + \alpha w^{n-1}zh^n) + (b + \alpha zh^n),
\end{align*}
\]

where \( \alpha^n = -1 \).

From (1), (2) and (3) we get \( b^n + z^n h^n \) and \( b^n - z^n h^n \in Hs(F_\alpha) \). Finally, since \( b^n = \frac{1}{2}(b^n + z^n h^n) + \frac{1}{2}(b^n - z^n h^n) \) and \( h \neq 0 \), then \( b^n \) cannot be an extreme point of \( Hs(F_\alpha) \). This contradiction completes the proof that \( Ext Hs(F_\alpha) \subseteq \{F_\alpha \circ \phi : \phi \in Ext B(H_0^\infty)\} \).

On the other hand, let \( \phi \in Ext B(H_0^\infty) \). Since \( F_\alpha \) is a finite Blaschke product, by \([9, \text{Theorem 9}]\) we have \( F_\alpha \circ \phi \) is an extreme point of \( B(H_\infty) \). Since \( Hs(F_\alpha) \subseteq B(H_\infty) \), we conclude that \( F_\alpha \circ \phi \) is an extreme point of \( Hs(F_\alpha) \). This completes the proof of Theorem 2.
3. An analogue of Ryff's theorem in Bergman spaces. In [10], Ryff proved the following remarkable result: Let \( F \in H^p \), \( 0 < p < \infty \), and \( f < F \). Then \( \|f\|_p = \|F\|_p \) if and only if \( f = F \circ \phi \) for some inner function \( \phi \) with \( \phi(0) = 0 \).

Concerning \( A^p \) spaces, Ryff's theorem is not true in general. For example, let \( F(z) = z \) and \( f(z) = z^2 \). Then \( f < F \) and it is easy to see that \( \|f\|_{A^p} \neq \|F\|_{A^p} \) for any \( p, 0 < p < \infty \). This suggests the following analogue of Ryff's theorem.

**Theorem 3.** Let \( F \) be a function in \( A^p \), \( 0 < p < \infty \), and \( f < F \). Then \( \|f\|_{A^p} = \|F\|_{A^p} \) if and only if \( f = F \circ \phi \), \( \phi(z) = cz \) for some \( c, |c| = 1 \).

**Proof.** Let \( \|f\|_{A^p} = \|F\|_{A^p} \). Thus

\[
\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, dr = \int_0^1 \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \, dr.
\]

For \( g \in A \) and \( 0 < p < \infty \), let \( m_p(g,r) = (1/2\pi) \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \). For \( 0 < r < 1 \), let \( h(r) = m_p(F,r) - m_p(f,r) \). By the Littlewood subordination theorem [3, p. 191], \( h \) is nonnegative on \((0,1)\), and clearly it is continuous there. This together with equation (1) implies that \( h \in L^1(0,1) \). Hence \( \int_0^1 h(r) \, dr = 0 \) and \( h \) is continuous on \((0,1)\) we get \( h(r) = 0 \) on \((0,1)\). Thus \( m_p(f,r) = m_p(F,r) \). Hence by [3, p. 191], \( f(z) = F(cz) \), \( |c| = 1 \), as required.

Conversely, let \( f(z) = F(cz) \), \( |c| = 1 \). Put \( c = \cos \alpha + i \sin \alpha \) and \( z = x + iy \). Then \( c = (x \cos \alpha - y \sin \alpha) + i(y \cos \alpha + x \sin \alpha) \). Let \( T: (x,y) \to (u,v) \), where \( u \) and \( v \) are the real and imaginary parts of \( cz \). Then the Jacobian of \( T \) (denoted by \( J_T \)) is \( c^2 \). By using change of variables, we get

\[
\int_D \int |F(x,y)|^p \, dx \, dy = \int_D \int |F(u,v)|^p \, |J_T| \, du \, dv.
\]

Thus \( \|F\|_{A^p} = \|f\|_{A^p} \). This completes the proof of Theorem 3.

In [4] (see [5, p. 124]) the following result is proved.

**Theorem A.** Let \( F \in H^p \) for some \( p, 1 < p < \infty \), and \( f \in Hs(F) \). If \( \|f\|_p = \|F\|_p \) then \( f \in \text{Ext } Hs(F) \).

The next result is the analogue of Theorem A in \( A^p \) spaces.

**Theorem 4.** Let \( F \in A^p \), \( 1 < p < \infty \), and \( f \in Hs(F) \). If \( \|f\|_{A^p} = \|F\|_{A^p} \) then \( f \in \text{Ext } Hs(F) \).

**Proof.** First, we show that if \( F \in A^p \) (\( p \geq 1 \)) then \( Hs(F) \subset A^p \), and moreover if \( f \in Hs(F) \) then \( \|f\|_{A^p} \leq \|F\|_{A^p} \). One way to see this is to use an argument similar to the one used in [4, Theorem 7]. So let \( f \in Hs(F) \). Then \( f \) can be approximated uniformly on compact subsets of \( D \), by functions of the form \( \sum_{k=1}^n \lambda_k f_k \), where \( \lambda_k \geq 0 \), \( \sum_{k=1}^n \lambda_k = 1 \) and \( f_k \in s(F) \). By the Littlewood subordination theorem we have \( m_p(f_k,r) \leq m_p(F,r) \) for \( 0 < r < 1 \) and \( k = 1,2,\ldots,n \). By the Minkowski inequality we get

\[
m_p \left( \sum_{k=1}^n \lambda_k f_k, r \right) \leq \sum_{k=1}^n \lambda_k m_p(f_k,r) \leq m_p(F,r).
\]

Consequently, \( m_p(f,r) \leq m_p(F,r) \) for \( 0 < r < 1 \). Thus we get

\[
\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, dr \leq \int_0^1 \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \, dr = \|F\|^p_{A^p}.
\]

Thus \( \|f\|_{A^p} \leq \|F\|_{A^p} \), as required.
Now, suppose that \( f = \lambda g + (1 - \lambda)h \), \( g, h \in Hs(F) \). Let \( f_1 = f/\|F\|_{A^p} \), \( g_1 = g/\|F\|_{A^p} \) and \( h_1 = h/\|F\|_{A^p} \). From above, \( \|g_1\|_{A^p} \leq 1 \) and \( \|h_1\|_{A^p} \leq 1 \). Since \( f_1 = \lambda g_1 + (1 - \lambda)h_1 \), \( \|f_1\|_{A^p} = 1 \) and \( A^p \) is a strictly convex space, we get \( g_1 = h_1 = f_1 \). Consequently, \( g = h = f \). This proves that \( f \in Ext Hs(F) \), and that ends the proof of the theorem.

In case \( F \in H^p \), \( 1 \leq p < \infty \), we have the following

**COROLLARY 5.** Let \( F \in H^p \), \( 1 \leq p < \infty \), and \( f \in Hs(F) \). If \( \|f\|_{A^r} = \|F\|_{A^r} \) for some \( r \), \( 1 < r \leq 2p \), then \( f \in s(F) \).

The proof of Corollary 5 requires the following known result.

**THEOREM B** [8, 11]. \( H^p \subset A^{2p} \), \( 0 < p < \infty \).

**PROOF OF COROLLARY 5.** Since \( F \in H^p \), then by the proof of Theorem 7 in [4] we have \( Hs(F) \subset H^p \). Thus by Theorem B, \( Hs(F) \subset A^{2p} \). If \( 1 < r \leq 2p \), then \( Hs(F) \subset A^r \). If \( f \in Hs(F) \) with \( \|f\|_{A^r} = \|F\|_{A^r} \), then by Theorem 4, \( f \) is an extreme point of \( Hs(F) \). Consequently, by [6] \( f \in s(F) \) and this completes the proof of Corollary 5.

**ADDED IN PROOF.** Recently the author obtained the following: If \( P \) is a polynomial and \( F \) is univalent then \( Ext Hs(P \circ F) \subset \{(P \circ F) \circ \phi : \phi \in B(H^p_0)\} \).

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