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ON THE RELATION BETWEEN C^*-ALGEBRAS OF FOLIATIONS AND THOSE OF THEIR COVERINGS

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ABSTRACT. By using the theory of groupoid equivalence of P. S. Muhly, J. N. Renault and D. P. Williams (cf. [**5**, **7**]), we identify the relation between the C^*-algebra of a foliated manifold and those of its regular covering foliations.

1. Introduction. Let $(M, \mathcal{F})$ be a foliated manifold. Let $\pi: \tilde{M} \to M$ be a regular covering. Then there is a unique covering foliation $(\tilde{M}, \tilde{\mathcal{F}})$ in which the leaves are exactly the preimage of the leaves in $(M, \mathcal{F})$. A natural question is to understand how the C^*-algebras $C^*(M, \mathcal{F})$ and $C^*(\tilde{M}, \tilde{\mathcal{F}})$ of the foliations are related [**2**]. An identification of such relations would, to some extent, reduce the classification of C^*-algebras of all the foliations of a manifold to that of the universal covering, which is often much more accessible. For instance, the classification of the C^*-algebras of all the ordinary foliations of the plane is known [**12**], while the same classification for all hyperbolic two-manifolds is a huge problem. In many cases using such a relation, the Connes conjecture [**2**] about K-theory of foliations may reduce to that of simply connected manifolds. (For instance, Example 9 below.)

J. Renault introduced the notion of "equivalence of groupoids" in [**7**]. More recently P. Muhly, D. Williams and Renault [**5**] showed that the C^*-algebras associated to equivalent groupoids are strongly Morita equivalent. Using this result, we determine precisely the relation between $C^*(M, \mathcal{F})$ and $C^*(\tilde{M}, \tilde{\mathcal{F}})$ in Theorem 4. Then we introduce the notion of covering groupoids and restate Theorem 4 in a more general situation (Theorem 7). We then indicate some applications with several examples. We follow the definitions and notation of [**5**].

We thank R. Zimmer for valuable discussions concerning homogeneous foliations. We also thank J. Renault and D. Williams for sending us a copy of [**5**].

2. The main theorem. Let $G(M, \mathcal{F})$ and $G(\tilde{M}, \tilde{\mathcal{F}})$ be the holonomy groupoids of the corresponding foliations. Assume both are Hausdorff. Let Γ be the covering group of $\pi: \tilde{M} \to M$. The Γ-action on $\tilde{M}$ can be naturally extended to $G(\tilde{M}, \tilde{\mathcal{F}})$ by sending $[\tilde{\gamma}]$ to $[\tilde{\gamma}g]$ for a representative path $\tilde{\gamma}$ in $\tilde{M}$ and $g \in \Gamma$. Therefore we have a transformation groupoid $G(\tilde{M}, \tilde{\mathcal{F}}) \times \Gamma$, with the composition law $(\tilde{\gamma}, g)(\tilde{\gamma}g, g') = (\tilde{\gamma}, gg')$. Let $G = G(M, \mathcal{F})$, $H = G(\tilde{M}, \tilde{\mathcal{F}}) \times \Gamma$, and $Z = G(\tilde{M}, \tilde{\mathcal{F}})$. Clearly, the unit spaces are $G^0 = M$ and $H^0 = \tilde{M}$. Define $\rho: Z \to G^0$ and $\sigma: Z \to H^0$ by $\rho(\tilde{\gamma}) = \pi(r(\tilde{\gamma}))$ and $\sigma(\tilde{\gamma}) = s(\tilde{\gamma})$. Both are continuous open maps. The space Z is

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then canonically a right $G(\tilde{M},\tilde{\mathcal{F}})$-space, and also a right Γ-space. Let

$$Z * H = \{(\tilde{\gamma}_1,(\tilde{\gamma}_2,g)) \in Z \times H \mid s(\tilde{\gamma}_1) = r(\tilde{\gamma}_2)g\}.$$

Then the map from $Z * H \to Z$ defined by sending $(\tilde{\gamma}_1,(\tilde{\gamma}_2,g))$ to $\tilde{\gamma}_1(\tilde{\gamma}_2 g)$ is continuous. This defines Z as a right H-space. The action of H on Z is free and proper, so Z is actually a principal H-space.

In order to make Z a left G-space, we need to examine more closely the leaf covering groups and the holonomy structures.

LEMMA 1. *Let $\pi\colon \tilde{M} \to M$ be a regular covering manifold with covering group Γ, $N \subset M$ a connected submanifold, and $\tilde{N}$ a connected component of $\pi^{-1}(N)$. Then the restriction of π to $\tilde{N}$ is also regular, with the covering group Γ_N being a subgroup of Γ.*

PROOF. The covering group $\Gamma = \pi_1(M)/\pi_1(\tilde{M})$ acts on $\pi^{-1}(N)$ freely, and properly discontinuously. Let $\Gamma_{\tilde{N}}$ be the subgroup of Γ that leaves $\tilde{N}$ invariant. Then $\Gamma_{\tilde{N}}$ acts on $\tilde{N}$ also freely, and properly discontinuously and $\tilde{N} \simeq N/\Gamma_{\tilde{N}}$. Q.E.D.

In particular, if $L_{\tilde{x}}$ is the leave in $(\tilde{M},\tilde{\mathcal{F}})$ containing $\tilde{x} \in \pi^{-1}(x)$, where $x \in M$, then $L_{\tilde{x}}$ is a regular cover of L_x. We denote the covering group by Γ_x. On the other hand, for each $x \in M$, there is a holonomy group G_x^x, and we have the holonomy group bundle $\{G_x^x\}$ over M. If x_1, x_2 are on the same leaf, then any path γ connecting x_1 and x_2 induces an isomorphism $\gamma^*\colon G_{x_1}^{x_1} \xrightarrow{\simeq} G_{x_2}^{x_2}$ by mapping $[\gamma_1]$ to $[\gamma\gamma_1\gamma^{-1}]$. As a local homeomorphism, the covering map π induces an embedding $\overline{\pi}_x^*\colon G_{\tilde{x}}^{\tilde{x}} \to G_x^x$ for each $\tilde{x} \in \pi^{-1}(x)$.

LEMMA 2. *The group $\overline{\pi}_x^*\{G_{\tilde{x}}^{\tilde{x}}\}$ is a normal subgroup of G_x^x. Equivalently, $\overline{\pi}_x^*(G_{\tilde{x}_1}^{\tilde{x}_1}) = \overline{\pi}_x^*(G_{\tilde{x}_2}^{\tilde{x}_2})$ for $\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)$ if $L_{\tilde{x}_1} = L_{\tilde{x}_2}$.*

PROOF. Let α $(\tilde{\alpha})$ be the quotient map from $\pi_1(L_x)$ $(\pi_1(L_{\tilde{x}}))$ onto G_x^x $(G_{\tilde{x}}^{\tilde{x}})$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(L_{\tilde{x}}) & \xhookrightarrow{\pi_x^*} & \pi_1(L_x) \\ {\scriptstyle\tilde{\alpha}}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G_{\tilde{x}}^{\tilde{x}} & \xhookrightarrow{\overline{\pi}_x^*} & G_x^x \end{array}$$

Let $\alpha(g)$ be any element in G_x^x and $\tilde{h} \in G_{\tilde{x}}^{\tilde{x}}$. Then a diagram chasing produces some $h \in \pi_1(L_{\tilde{x}})$ such that $\alpha\pi_x^*(h) = \overline{\pi}_x^*(\tilde{h})$. There is $h_1 \in \pi_1(L_{\tilde{x}})$ such that $\pi_x^*(h_1) = g\pi_x^*(h)g^{-1}$ by Lemma 1. Thus $\alpha(g)\overline{\pi}_x^*(\tilde{h})\alpha(g^{-1}) = \overline{\pi}_x^*\tilde{\alpha}(h_1)$. Q.E.D.

Thus we may form the *quotient holonomy group bundle* $\{G_x^x/G_{\tilde{x}}^{\tilde{x}}\}$ over M. There is an obvious group homomorphism $\phi_x\colon \Gamma_x \to G_x^x/G_{\tilde{x}}^{\tilde{x}}$ defined as follows. An element $g \in \Gamma_x$ corresponds to a point $x_g \in \pi^{-1}(x) \cap L_{\tilde{x}}$ if we fix $\tilde{x}$ corresponding to the unit e. A path $\tilde{\gamma}_g$ starting at $\tilde{x}$ and ending at x_g gives a loop $\pi(\tilde{\gamma}_g)$ in M representing an element $\phi_x(g)$ in G_x^x, whose class in $G_x^x/G_{\tilde{x}}^{\tilde{x}}$ is uniquely defined by g. Given any $[\gamma]$ in G_x^x, there is a preimage $\tilde{\gamma}$ in $L_{\tilde{x}}$ starting at $\tilde{x}$. The point $r(\tilde{\gamma}) \in \pi^{-1}(x) \cap L_{\tilde{x}}$ corresponding to some $g \in \Gamma_x$. So ϕ_x is onto.

DEFINITION 3. The covering map $\pi\colon (\tilde{M},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ *of foliations* is said to be *regular* if the map ϕ is an isomorphism from the leaf covering group bundle to the quotient holonomy group bundle.

By the preceding discussion ϕ is an isomorphism if and only if ϕ_x is injective for all $x \in M$. The following is the main theorem.

THEOREM 4. *Let $\pi\colon \tilde{M} \to M$ be a regular covering with covering group Γ. If $\pi\colon (\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})$ is also a regular covering map of foliations, then the C^*-algebras $C^*(M, \mathcal{F})$ and $C^*(\tilde{M}, \tilde{\mathcal{F}}) \rtimes \Gamma$ are isomorphic, where the Γ-action is induced from that on the holonomy groupoid $G(\tilde{M}, \tilde{\mathcal{F}})$.*

LEMMA 5. *Assume $\pi\colon (\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})$ is regular. Then $Z = G(\tilde{M}, \tilde{\mathcal{F}})$ is naturally a left principal $G(M, \mathcal{F})$-space.*

PROOF. Let $[\gamma] \in G(M, \mathcal{F})$ and $[\tilde{\gamma}_1] \in G(\tilde{M}, \tilde{\mathcal{F}})$, such that $\pi(r(\tilde{\gamma}_1)) = s(\gamma)$. There is a unique path $\tilde{\gamma}$ such that $\pi(\tilde{\gamma}) = \gamma$ and $s(\tilde{\gamma}) = r(\tilde{\gamma}_1)$. We define a composition by $[\gamma][\tilde{\gamma}_1] = [\tilde{\gamma}\,\tilde{\gamma}_1]$. It is easy to check that a particular choice of γ and $\tilde{\gamma}_1$ is irrelevant. This is clear for $\tilde{\gamma}_1$. Assume $[\gamma'] = [\gamma]$, $\pi(\tilde{\gamma}') = \gamma'$, and $s(\tilde{\gamma}') = r(\tilde{\gamma}_1)$. Denote $x = r(\gamma)$ and $\tilde{x} = r(\tilde{\gamma})$. Then $\pi(\tilde{\gamma}'\tilde{\gamma}^{-1}) = \gamma'\gamma^{-1} \sim 0$ in G_x^x. Together with the assumption $\Gamma_x \simeq G_x^x/G_{\tilde{x}}^{\tilde{x}}$, this implies that $r(\tilde{\gamma}') = \tilde{x}$ and $[\tilde{\gamma}] = [\tilde{\gamma}']$, as $G_{\tilde{x}}^{\tilde{x}}$ is included in G_x^x.

The G-action on Z is then automatically free. If $[\tilde{\gamma}\,\tilde{\gamma}_1] = [\tilde{\gamma}_1]$, then $[\tilde{\gamma}]$ is a unit and $[\gamma]$ is a unit. Similarly the map $G * Z \to Z$ is proper since the left action of $G(\tilde{M}, \tilde{\mathcal{F}})$ on itself is proper. Q.E.D.

PROOF OF THEOREM 4. We need to check that our choice of G, H and Z yields a groupoid equivalence [**5**, Definition 2.1] and then apply [**5**, Theorem 2.8].

We have shown that Z is a left principal G-space (Lemma 5) and a right principal H-space. The G-action and the right $G(\tilde{M}, \tilde{\mathcal{F}})$-action on Z commute because of the associativity of groupoid composition. If $[\gamma] \in G$, $[\tilde{\gamma}], [\tilde{\gamma}_1] \in Z$ are such that $s(\tilde{\gamma}) = r(\tilde{\gamma}_1)$, $\pi(\tilde{\gamma}) = \gamma$, then for any $g \in \Gamma$, we have

$$[\gamma]([\tilde{\gamma}_1]g) = [\gamma][\tilde{\gamma}_1 g] = [(\tilde{\gamma}g)(\tilde{\gamma}_1 g)] = [(\tilde{\gamma}\tilde{\gamma}_1)g] = ([\gamma] \cdot [\tilde{\gamma}_1])g.$$

Thus the left G-action and the right Γ-action commute. For each $m \in \tilde{M} = H^0$, the preimage $\sigma^{-1}(m)$ is exactly the leaf containing m in $(\tilde{M}, \tilde{\mathcal{F}})$. Thus the map σ induces a homeomorphism from $G\backslash Z = G(M, \mathcal{F})\backslash G(\tilde{M}, \tilde{\mathcal{F}})$ onto H^0. Similarly, the map $\rho\colon Z \to G^0$ induces a homeomorphism from $Z/H = G(\tilde{M}, \tilde{\mathcal{F}})/\{G(\tilde{M}, \tilde{\mathcal{F}}) \times \Gamma\} = \tilde{M}/\Gamma$ onto M. Q.E.D.

Undoubtedly Theorem 4 can also be proved directly with strong Morita equivalence by finding a suitable imprimitivity bimodule (which is $C_c(G(\tilde{M}, \tilde{\mathcal{F}}))$ from our choice of Z). However the Muhly-Renault-Williams theorem provides us a particularly convenient general set-up which greatly simplifies the argument. This seems to verify their prediction in the introduction of [**5**]. Of course, the approach of strong Morita equivalence (introduced by M. Rieffel, cf. [**1**]) is crucial, both here and in [**4, 5**].

We now generalize Theorem 4 to the setting of abstract groupoids. Let G, H be second countable locally compact groupoids with Haar systems λ and μ, respectively. Assume that both unit spaces G^0 and H^0 are paracompact and arcwise connected. Suppose that there is a regular covering map $\pi\colon G^0 \to H^0$ with covering group Γ, and that the orbit O_x is connected for every $x \in H^0$. Let Γ_x be the covering group of an orbit $O_{\tilde{x}}$ over O_x, where $\tilde{x} \in \pi^{-1}(x)$. Then there is again an embedding $\pi_x^*\colon G_{\tilde{x}}^{\tilde{x}} \hookrightarrow G_x^x$ of isotropy groups and a group epimorphism $\phi_x\colon \Gamma_x \twoheadrightarrow G_x^x/G_{\tilde{x}}^{\tilde{x}}$.

DEFINITION 6. With G and H given as above, we say that G is a *regular covering groupoid* of H if the maps ϕ_x are injective for all $x \in H^0$.

Analogous to Theorem 4, we have

THEOREM 7. *Let G be a regular covering groupoid of H with covering group Γ. Then $C^*(G,\lambda) \rtimes \Gamma$ is strongly Morita equivalent to $C^*(H,\mu)$.*

3. Applications.

EXAMPLE 8. The condition that the covering map $\pi\colon \tilde{M} \to M$ is regular does not automatically imply that the map $\pi\colon (\tilde{M},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ is regular. Consider the simplest example. Let $\tilde{M} = \mathbf{R}^2$ and $M = \mathbf{R} \times S^1$. Then the projection $\pi\colon \tilde{M} \to M$ is regular. Let $\tilde{\mathcal{F}}$ be the foliation by parallel lines with a fixed angle α to the horizontal direction, $0 \le \alpha \le \pi/2$. Then $\pi\colon (\tilde{M},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ is regular unless $\alpha = \pi/2$, when $(M,\mathcal{F})$ is foliated by parallel S^1. Of course then $C^*(\tilde{M},\tilde{\mathcal{F}}) \cong C^*(M,\mathcal{F}) = C_0(\mathbf{R}) \otimes \mathcal{K}$. We see that the regular coverings of foliations are "generic" in the sense that the deck transformations take a general position with respect to $\tilde{\mathcal{F}}$. This fact will become clearer in Example 10.

EXAMPLE 9. Let $(M,\mathcal{F})$ be an annular with Reeb foliation. Let $(\tilde{M},\tilde{\mathcal{F}})$ be its universal covering foliation. Then [**12**, Theorem 4.1.2] tells us that $C^*(\tilde{M},\tilde{\mathcal{F}})$ is isomorphic to

$$A_2 = \left\{ f \in C_0([0,1), M_2(\mathcal{K})) \mid f(0) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

One verifies that $\pi\colon (\tilde{M},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ is regular, and Theorem 7 yields that $C^*(M,\mathcal{F})$ is isomorphic to $A_2 \rtimes \mathbf{Z}$ by the well-known facts that foliation C^*-algebras are stable [**4**] and strongly Morita equivalent stable C^*-algebras are isomorphic [**1**]. (For a constructive proof, see [**14**].) This explicit characterization of C^*-algebras of foliated Reeb components provides an alternative proof of the results of A. M. Torpe [**9**].

EXAMPLE 10. Both situations described in Examples 8 and 9 are special cases of the following [**11**]. Let Σ be a closed two-manifold and $\mathcal{F}$ a structurely stable flow on Σ. Let $M = \Sigma \setminus \{\text{singularities}\}$. Then the universal covering manifold $\tilde{M} = \mathbf{R}^2$ and $\pi\colon (\tilde{M},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ is regular with the fundamental group Γ of M being a free group of finitely many generators. The structure of $C^*(\tilde{M},\tilde{\mathcal{F}})$ is given in Theorem 4.3.1 of [**12**], which shows that these C^*-algebras are classified by distinguished trees. Thus, applying Theorem 4, we get a characterization of $C^*(\tilde{M},\tilde{\mathcal{F}})$.

More generally, let $(M,\mathcal{F})$ be any foliated hyperbolic two-manifold with corresponding Fuschian group Γ and the covering foliation $(\mathbf{H},\tilde{\mathcal{F}})$, where $\mathbf{H}$ is the hyperbolic upper half-plane. If the covering map $\pi\colon (\mathbf{H},\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ is regular, then we can characterize $C^*(M,\mathcal{F})$ by [**12**] and Theorem 4. The same method actually applies to even more general situations. Let Γ be a lattice in a connected semisimple Lie group. Suppose that the homogeneous space $M = \Gamma \setminus G/K$ admits a foliation $\mathcal{F}$, then often we can describe $C^*(M,\mathcal{F})$ in terms of $C^*(G/K,\tilde{\mathcal{F}})$. For instance, in light of Thurston's program, we can hopefully reduce the study of C^*-algebras of foliations of any hyperbolic three-manifolds to that of $\mathbf{H}^3$ (or $\mathbf{H}^2 \times \mathbf{R}$) and actions of three-manifold groups.

Let G be a connected Lie group, with a connected subgroup H and a lattice Γ. Let $\mathcal{F}$ be the foliation of $\Gamma \setminus G$ by the cosets of H. From the proofs of Theorem 4 and Proposition 1.5 of Part I of [**10**] we have

PROPOSITION 11. *If the regularity condition (Definition 3) of the covering $G \to \Gamma \setminus G$ holds, then the C^*-algebra $C^*(\Gamma \setminus G, \mathcal{F})$ is isomorphic to the cross product $C_0(\Gamma \setminus G) \rtimes H$.*

The following observations are due to Professor R. Zimmer. See [**13**] for relevant results in Lie foliations.

LEMMA 12. *Let H_g be the algebraic hull of $g\Gamma g^{-1} \cap H$ with Lie algebra $\mathfrak{h}_g$, for $[g] \in G/H$. Then the regularity condition holds if the induced adjoint representation of H_g on the quotient $\mathfrak{g}/\mathfrak{h}$ of the corresponding Lie algebra is faithful.*

EXAMPLE 13. Consider the horocycle flow where G is $\mathrm{SL}(2;\mathbf{R})$, H is the 1-parameter nilpotent subgroup N consisting of strictly upper triangular matrices. Then the regularity condition holds. It follows from Theorem 4 that $C^*(\Gamma \setminus G,$ horocycle$) \simeq C_0(S^1 \times \mathbf{R}) \rtimes \Gamma$, since $G/N \simeq S^1 \times \mathbf{R}$. To see the faithfulness of Ad_N, we recall the equality

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = -2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the general situation, where G is a semisimple Lie group and H is an arbitrary nilpotent subgroup, the regularity condition still holds. In fact any nilpotent element x is contained in a copy of $\mathrm{SL}(2,\mathbf{R})$ in G and the argument above shows that the induced action of ad_x on $\mathfrak{g}/\mathfrak{h}$ is nonzero.

EXAMPLE 14. As an example showing that the regularity condition may fail for general solvable subgroups in semisimple Lie groups, we consider again $G = \mathrm{SL}(2,\mathbf{R})$ and $\Gamma = \mathrm{SL}(2,\mathbf{Z})$, where H is the $\{ax+b\}$ solvable subgroup consisting of the upper triangular matrices. Then $\mathrm{ad}_x = 0$ for $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in the Lie algebra $\mathfrak{h}$.

However the failure of the regularity condition is only an exceptional rather than general phenomenon, as we observed earlier in Examples 8 and 10.

EXAMPLE 15. Let G be a semisimple Lie group of rank $r \geq 2$ with trivial center. Let $\mathcal{F}$ be the homogeneous foliation on $\Gamma \setminus G$ induced by a solvable subgroup H of G. Then the regularity condition holds.

Without loss of generality, we may assume G is simple. The adjoint representation of an abelian subalgebra of $\mathfrak{h}$ is clearly faithful. Take a nilpotent generator $x \in \mathfrak{h}$. We may consider $\mathfrak{h}$ as contained in the upper triangular matrices and identify x as an elementary matrix E_{ij}, with 1 in the i,jth entry and 0 elsewhere, $i < j$. Since $r \geq 2$, there is l distinct from i and j, so that $[E_{ij}, E_{li}] = -E_{lj}$.

EXAMPLE 16. Let G again be semisimple, but H an abelian subgroup of G with $H \cap Z(G) = 0$. (For $G = \mathrm{SL}(2,\mathbf{R})$ we get in particular the geodesic flows on $\Gamma \setminus G$.) Again the regularity condition holds. To see this, let $\overline{H}$ be the algebraic hull of H. Then $\overline{H}$ is an abelian algebraic group. It suffices to see that the adjoint representation of $\overline{H}$ is faithful on $\mathfrak{g}/\overline{\mathfrak{h}}$. There is a unique decomposition $\overline{H} = T \times N$, where T is a torus (i.e., consisting of algebraic semisimple elements) and N is nilpotent. Suppose $g \in \overline{H}$ acts trivially. Let $g = tn$, so that t and n act trivially. This implies immediately that $t \in Z(G) \cap H = 0$. Since n acts trivially, so does

X_n, a corresponding nilpotent generator. Again there is a Lie subalgebra $\mathfrak{h}_2$ of $\mathfrak{g}$ containing X_n that is isomorphic to $\mathrm{Sl}(2, \mathbf{R})$. There is some $\overline{X}_n \in \mathfrak{h}_2$ such that $[X_n, \overline{X}_n] = A$, which is semisimple, does not commute with X_u.

We conclude with a problem: Determine if the following converse of the Muhly-Renault-Williams theorem is true. If both G and H are locally compact principal groupoids with Haar systems λ, μ such that $C^*(G, \lambda)$ and $C^*(H, \mu)$ are isomorphic then there is a groupoid equivalence Z between G and H.

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