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L^p MULTIPLIERS; A NEW PROOF OF AN OLD THEOREM

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ABSTRACT. New proofs are given for the following results of Hirschman and Wainger: Let $\psi \in C^\infty(\mathbf{R}^n)$ vanish in a neighborhood of the origin; $\psi(\xi) = 1$ for large ξ. Then

$$|\xi|^{-\beta}\psi(\xi)\exp(i|\xi|^\alpha)$$

is a multiplier in $L^p(\mathbf{R}^n)$ for $|1/p - 1/2| < \beta/n\alpha$; is not a multiplier in $L^p(\mathbf{R}^n)$ for $|1/p - 1/2| > \beta/n\alpha$.

1. Introduction. Let α, β be real numbers, $0 < \alpha < 1$, $\beta > 0$. Let $\psi \in C^\infty(\mathbf{R}^n)$ be 0 near the origin and 1 outside a compact subset of $\mathbf{R}^n$. For $f \in S(\mathbf{R}^n)$ we define $Tf \in S(\mathbf{R}^n)$ by

$$[Tf]\hat{}(\xi) = |\xi|^{-\beta}\psi(\xi)\exp(i|\xi|^\alpha)\hat{f}(\xi). \tag{1}$$

T is then a strongly singular convolution operator. The kernel of T was first studied by G. H. Hardy for $n = 1$ (cf. [**2**]). In [**3**], I. I. Hirschman develops the L^p-theory for operators of type T, concentrating in the periodic case. He proves, for $n = 1$ (see [**3**, Theorem 3c and remarks following it]),

THEOREM 1. *Let $|1/p - 1/2| < \beta/n\alpha$. Then T extends to a bounded operator on L^p.*

THEOREM 2. *T does not extend to a bounded operator on L^p if $|1/p - 1/2| > \beta/n\alpha$.*

The proof of Theorems 1, 2 for general dimension n is due to S. Wainger [**7**] and E. Stein [**5**]. In [**1**], C. Fefferman picks up the problem by looking at the behavior of T in the limit case $1/p - 1/2 = \beta/n\alpha$. He proves that in this case T is somewhat better than of weak type (p,p), thus proving Theorem 1 by the Marcinkiewicz interpolation theorem (cf., for example, [**6**]) and duality. His paper is also the first one to give information about behavior in the limit cases.

We will give new proofs of Theorems 1, 2 which we believe are simpler and more direct than the previous ones. In the sequel, $\|\cdot\|_p$ denotes the norm of $L^p(\mathbf{R}^n)$; $\mathbf{B}(L^p)$ is the space of bounded operators on $L^p = L^p(\mathbf{R}^n)$; α, β, ψ are as described above; $0 < \delta < \rho$ are such that $\psi(\xi) = 0$ if $|\xi| \le \delta$ and $\psi(\xi) = 1$ if $|\xi| \ge \rho$. All integrals without an explicit domain of integration are over all of $\mathbf{R}^n$.

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2. Proof of Theorem 1. We will need the following lemma.

LEMMA 1. *For $\sigma > 0$ define k_σ on $\mathbf{R}^n$ by*

$$k_\sigma(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi}\psi(\xi)\exp((i-\sigma)|\xi|^\alpha)\,d\xi. \tag{2}$$

Then there exists a constant C such that

$$\|k_\sigma\|_1 \le C\sigma^{-n/2}\exp(-\tfrac{1}{4}\delta^\alpha\sigma) \tag{3}$$

for all $\sigma > 0$.

We postpone the proof of Lemma 1 to §4. Let T be given by (1),

$$Tf(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi}\psi(\xi)|\xi|^{-\beta}\exp(i|\xi|^\alpha)\hat{f}(\xi)\,d\xi.$$

Noticing that

$$|\xi|^{-\beta} = \frac{1}{\Gamma(\beta/\alpha)}\int_0^\infty \sigma^{(\beta/\alpha)-1}\exp(-\sigma|\xi|^\alpha)\,d\sigma,$$

we can write

$$Tf(x) = \frac{1}{\Gamma(\beta/\alpha)}\int_0^\infty \sigma^{(\beta/\alpha)-1}(k_\sigma * f)(x)\,d\sigma,$$

hence

$$\|Tf\|_p \le \frac{1}{\Gamma(\beta/\alpha)}\int_0^\infty \sigma^{(\beta/\alpha)-1}\|k_\sigma * f\|_p\,d\sigma \tag{4}$$

for all $p \in [1,\infty]$. By Lemma 1,

$$\|k_\sigma * f\|_1 \le C\sigma^{-n/2}\exp(-\tfrac{1}{4}\delta^\alpha\sigma)\|f\|_1; \tag{5}$$

since $|\hat{k}_\sigma(\xi)| \le \|\psi\|_\infty\exp(-\frac{1}{4}\delta^\alpha\sigma)$,

$$\|k_\sigma * f\|_2 \le \|\psi\|_\infty\exp(-\tfrac{1}{4}\delta^\alpha\sigma)\|f\|_2. \tag{6}$$

By the Riesz-Thorin interpolation theorem (cf. [**8**]) (5) and (6) imply for $1 \le p \le 2$,

$$\|k_\sigma * f\|_p \le C_1\sigma^{-\lambda}\exp(-\nu\delta^\alpha\sigma)\|f\|_p \tag{7}$$

for $\sigma > 0$, where C_1 depends only on the C of (5) and $\|\psi\|_\infty$, $\lambda = n((1/p)-(1/2))$ and $\nu = (1/p - 1/2)/2 + 2(1-1/p) > 0$. Using (7) in (4), we see $T \in \mathbf{B}(L^p)$ if

$$\int_0^\infty \sigma^{(\beta/\alpha)-1-\lambda}\exp(-\nu\delta^\alpha\sigma)\,d\sigma < \infty,$$

which happens if (and only if) $\lambda < \beta/\alpha$; i.e., $1/p - 1/2 < \beta/n\alpha$. This proves Theorem 1 if $1 < p < 2$; the case $p \ge 2$ follows by duality.

3. Proof of Theorem 2. We use the following result from stationary phase analysis.

LEMMA 2. *Let $P \in C^\infty(\mathbf{R}^n)$, $g \in C_0^\infty(\mathbf{R}^n)$ and assume* $\det[(\partial^2/\partial\xi_i\partial\xi_j)P(\xi)] \neq 0$ *for all $\xi \in \operatorname{supp} g$. Then there exists $C = C(g)$ such that*

$$\left| \int \exp(ix\cdot\xi - itP(\xi))g(\xi)\,d\xi \right| \leq Ct^{-n/2}$$

for all $t > 0$.

For a proof of Lemma 2, see (for example) [**4**, p. 41].

For $t > 0$, we define

$$T(t)f(x) = (2\pi)^{-n/2}\int \exp(ix\cdot\xi + it|\xi|^\alpha)|\xi|^{-\beta}\psi(t^{1/\alpha}\xi)\hat{f}(\xi)\,d\xi \tag{8}$$

so that $T = T(1)$. Let E be the set of all $f \in (\mathbf{R}^n)$ such that $\hat{f} \in C_0^\infty$ and $\operatorname{supp} f \subseteq \{\xi \mid |\xi| > \rho\}$. Noticing $\psi(t^{1/\alpha}\xi)\hat{f}(\xi) = \hat{f}(\xi)$ for $f \in E$, $t \geq 1$, all ξ; we can apply Lemma 2 with $P(\xi) = |\xi|^\alpha$ for $|\xi| \geq \rho$; $g(\xi) = |\xi|^{-\beta}\hat{f}(\xi)$. The determinant in the lemma evaluates to $(-\alpha|\xi|^{\alpha-2})^n(1-\alpha) \neq 0$ for $0 < \alpha < 1$, $|\xi| \geq \rho$. Thus

$$\|T(t)f\|_\infty \leq C(f)t^{-n/2} \tag{9}$$

for $f \in F$, $t \geq 1$. Assume now $1 \leq p \leq 2$ and $T \in \mathbf{B}(L^p)$, of norm C_p. It is then easy to see $T(t) \in \mathbf{B}(L^p)$ for all $t > 0$ and

$$\|T(t)f\|_p \leq C_p t^{\beta/\alpha}\|f\|_p \tag{10}$$

for all $f \in L^p$. From (10) and (9) we get at once

$$\|T(t)f\|_2 \leq Kt^\lambda \tag{11}$$

by a trivial interpolation, K depending on f and $\lambda = (p/2n)[\beta/(n\alpha) - (1/p - 1/2)]$, $f \in E$. Since $\|T(t)f\|_2 = \|\hat{f}/|\xi|^\beta\|_2$ for $t \geq 1$, i.e., is independent of t for $t \geq 1$ and $\neq 0$ if $f \in E\backslash\{0\}$, (11) can only hold if $\lambda \geq 0$, i.e., $1/p - 1/2 \leq \beta/n\alpha$. This proves Theorem 2 in case $1 \leq p \leq 2$; the case $p \geq 2$ follows by duality.

4. Proof of Lemma 1. For $r = 0, 1, 2, \ldots$, let $k_\sigma^{(r)} = (\partial/\partial\sigma)^r k_\sigma$; thus

$$[k_\sigma^{(r)}]^\frown(\xi) = (-1)^r\psi(\xi)|\xi|^{r\alpha}\exp((i-\sigma)|\xi|^\alpha).$$

If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index of length m, then

$$|D_\xi^\gamma k_\sigma^{(r)}(\xi)| \leq C[\chi_S(\xi)\exp(-\tfrac{1}{2}\delta^\alpha\sigma) + |\xi|^{r\alpha - m(1-\alpha)}\exp(-\tfrac{1}{2}|\xi|^\alpha\sigma)]\exp(-\tfrac{1}{4}\delta^\alpha\sigma) \tag{12}$$

for $|\xi| \geq \delta$, $\sigma > 0$, where χ_S is the characteristic function of $S = \{\xi | \delta \leq |\xi| \leq \rho\}$ and C depends on ψ, m, r. The proof is straightforward, though tedious. Terms in which ψ is differentiated can be estimated by const $\chi_S\exp(-|\xi|^\alpha\sigma)(1+\sigma)^m$; in the other terms we can estimate (for $|\xi| \geq \delta$) all powers of ξ by the highest appearing power, namely $|\xi|^{r\alpha - m(1-\alpha)}$. We get rid of all positive powers of $1 + \sigma$ by using part of $\exp(-|\xi|^\alpha\sigma) \leq \exp(-\delta^\alpha\sigma)$ for that purpose. Another part of this factor gets factored out at the end. From (12) we get

$$\|\,|x|^m k_\sigma^{(r)}\|_2 \leq C\left[\exp\left(-\frac{1}{2}\delta^\alpha\sigma\right) + \sigma^{\lambda-(n/2\alpha)}\left(\int g(\xi)\,d\xi\right)^{1/2}\right]\exp\left(-\frac{1}{4}\delta^\alpha\sigma\right) \tag{13}$$

where

$$g(\xi) = |\xi|^{-2\lambda\alpha} \exp(-|\xi|^\alpha), \qquad \lambda = \frac{m(1-\alpha)}{\alpha} - r.$$

We see $g \in L^1(\mathbf{R}^n)$ iff $2\lambda\alpha < n$, i.e.,

$$m < (n + 2r\alpha)/(2(1-\alpha)). \tag{14}$$

Thus (13) implies

$$\| \, |x|^m k_\sigma^{(r)} \|_2 \leq C\sigma^{m(1-\alpha)/\alpha - (r + n/2\alpha)} \exp(-\tfrac{1}{4}\delta^\alpha \sigma) \tag{15}$$

for all integers r, m such that (14) holds, $\sigma > 0$. In particular, (15) holds for $m = 0$, all $r = 0, 1, \ldots$. Now choose and fix an integer $m > n/2$; then fix an integer $r \geq 0$ such that (14) holds. Using

$$\|h\|_1 \leq \text{const}\, R^{n/2}(\|h\|_2 + R^{-m} \| \, |x|^m h\|_2),$$

valid for all $R > 0$, we get from (15) with $h = k_\sigma^{(r)}$, $R = \sigma^{(1-\alpha)/\alpha}$,

$$\|k_\sigma^{(r)}\|_1 \leq C\sigma^{-(n/2)-r} \exp(-\tfrac{1}{4}\delta^\alpha \sigma)$$

for $\sigma > 0$. Integrating r times with respect to σ, from σ to ∞, we obtain (3).

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