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STRONGLY COMMUTING SELFADJOINT OPERATORS AND COMMUTANTS OF UNBOUNDED OPERATOR ALGEBRAS

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ABSTRACT. Let A_1 and A_2 be (unbounded) selfadjoint operators on a Hilbert space $\mathcal{H}$ which commute on a dense linear subspace of $\mathcal{H}$. To conclude that A_1 and A_2 strongly commute, additional assumptions are necessary. Two propositions which contain such additional conditions are proved in §1. In §2 we define different commutants of unbounded operator algebras (form commutant, weak unbounded commutant, strong unbounded commutant) and we discuss the relations between them and their bounded parts. In §3 we construct a selfadjoint $*$-representation of the polynomial algebra in two variables for which the form commutant is different from the weak unbounded commutant.

1. Conditions for strong commutativity of selfadjoint operators. Let $\mathcal{H}$ be a complex Hilbert space. As usual, we say that a bounded operator C and a possibly unbounded operator T on H *commute* if $CT \subseteq TC$. We say that two selfadjoint operators A_1 and A_2 on $\mathcal{H}$ *strongly commute* if the spectral projections of A_1 and A_2 mutually commute. This is the case if and only if $(A+z)^{-1}$ and A_2 commute for some (and then for all) $z \in \mathbf{C}\backslash\mathbf{R}$.

In Propositions 2 and 3 and in Lemma 1 we assume the following. Let B_1 and B_2 be symmetric linear operators and let A be a selfadjoint operator acting on the same Hilbert space $\mathcal{H}$ such that $\mathcal{D}(B_1) \subseteq \mathcal{D}(A)$ and $\mathcal{D}(B_2) \subseteq \mathcal{D}(A)$. We assume that there exists a constant λ such that

$$\|B_l\varphi\| \leq \lambda\|(A+i)\varphi\| \quad \text{for } l = 1,2 \text{ and for all } \varphi \in \mathcal{D}(B_1). \tag{1}$$

Upon the formulation, the following Proposition 2 is due to N. S. Poulsen [**5**, Lemma 2]. The author is indebted to P. E. T. Jørgensen for this information. Poulsen's proof is essentially based on Nelson's theorem on analytic domination [**4**, Theorem 8]. Our proof given below uses only elementary operator theory, so it seems to be of some interest in itself. The following lemma is used in the proofs of Propositions 2 and 3 below.

LEMMA 1. *Suppose $l \in \{1,2\}$. Suppose that there is a linear subspace $\mathcal{D}_1 \subseteq \mathcal{D}(AB_l) \cap \mathcal{D}(B_lA)$ such that $AB_l\varphi = B_lA\varphi$ for $\varphi \in \mathcal{D}_1$ and such that $\mathcal{D}_1$ is a core for A.*

Then, $\overline{B_l}$ is selfadjoint and $\overline{B_l}$ and A strongly commute.

PROOF. From assumption (1) it follows that there is a bounded operator X_l on $\mathcal{H}$ such that $B_l\varphi = X_l(A+i)\varphi$ for $\varphi \in \mathcal{D}(B_l)$. Since $\mathcal{D}_l$ and so $\mathcal{B}(B_l)$ is a core for

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A and hence for $X_l(A+i)$, we get

$$B_l^* = (X_l(A+i) \upharpoonright \mathcal{D}(B_l))^* = (X_l(A+i))^* = (A-i)X_l^*.$$

Putting $Y_l := (A-i)|A-i|^{-1}X_l^*$, we therefore have

$$B_l^* = |A-i|Y_l. \tag{2}$$

Since $Y_l^* = X_l(A+i)|A-i|^{-1}, B_l = X_l(A+i) = Y_l^*|A-i|$ on $\mathcal{D}(B_l)$. That is, since B_l is symmetric, we have

$$B_l = |A-i|Y_l = Y_l^*|A-i| \quad \text{on } \mathcal{D}(B_l). \tag{3}$$

For $\varphi \in \mathcal{D}_l$,

$$\begin{aligned}(A+i)B_l\varphi = (A+i)|A-i|Y_l\varphi &= |A-i|(A+i)Y_l\varphi \\ = B_l(A+i)\varphi &= |A-i|Y_l(A+i)\varphi\end{aligned}$$

and so $(A+i)Y_l\varphi = Y_l(A+i)\varphi$. Since $\mathcal{D}_l$ is a core for A, $(A+i)\mathcal{D}_l$ is dense in $\mathcal{H}$. Hence Y_l commutes with $(A+i)^{-1}$. Consequently, Y_l^* commutes with $(A-i)^{-1}$ and so with $|A-i|$. Putting this into (3), we get $B_l = Y_l^*|A-i| = |A-i|Y_l^* = |A-i|Y_l$ on $\mathcal{D}(B_l)$. Since $\mathcal{D}(B_l)$ is dense in $\mathcal{H}$, $Y_l = Y_l^*$.

We prove that $\overline{B_l}$ is selfadjoint. For suppose that $B_l^*\varphi = z\varphi$ for some $\varphi \in \mathcal{H}$ and for $z = i$ or $z = -i$. Then, from (2), $Y_l\varphi = z|A-i|^{-1}\varphi$. Thus

$$\langle Y_l\varphi, \varphi\rangle = z\langle |A-i|^{-1}\varphi, \varphi\rangle = z\| \, |A-i|^{-1/2}\varphi\|.$$

Since $Y_l = Y_l^*$, $\langle Y_l\varphi, \varphi\rangle$ is real and so $|A-i|^{-1/2}\varphi = 0$. Hence $\varphi = 0$. This shows that $\overline{B_l}$ is selfadjoint.

Since $\overline{B_l} = B_l^*$ as just shown, (2) gives $\overline{B_l} = |A-i|Y_l$. Since $(A+i)^{-1}$ commutes with Y_l, the latter yields

$$(A+i)^{-1}\overline{B_l} = (A+i)^{-1}|A-i|Y_l = |A-i|Y_l(A-i)^{-1} = \overline{B}_l(A+i)^{-1} \quad \text{on } \mathcal{D}(\overline{B_l}),$$

i.e., $\overline{B_l}$ and A strongly commute. □

PROPOSITION 2. *Suppose that there are linear subspaces $\mathcal{D}_l \subseteq \mathcal{D}(AB_l) \cap \mathcal{D}(B_lA)$ for $l = 1,2$ and $\mathcal{D}_{12} \subseteq \mathcal{D}(B_1B_2) \cap \mathcal{D}(B_2B_1)$ such that $AB_l\varphi = B_lA\varphi$ for $\varphi \in D_l$, $B_1B_2\psi = B_2B_1\psi$ for $\psi \in \mathcal{D}_{12}$ and such that $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_{12}$ are cores for A.*

Then $\overline{B_1}$ and $\overline{B_2}$ are strongly commuting selfadjoint operators.

PROOF. Because of Lemma 1, it only remains to show that the selfadjoint operators $\overline{B_1}$ and $\overline{B_2}$ strongly commute. To prove this, we use the notation and the facts established in the proof of Lemma 1. By (3),

$$B_1B_2\psi = |A-i|Y_1|A-i|Y_2\psi = B_2B_1\psi = |A-i|Y_2|A-i|Y_1\psi$$

for $\psi \in \mathcal{D}_{12}$. Since Y_1 and Y_2 both commute with A and $\ker|A-i| = \{0\}$, this implies that $Y_1Y_2|A-i|\psi = Y_2Y_1|A-i|\psi$ for $\psi \in \mathcal{D}_{12}$. Because $\mathcal{D}_{12}$ is a core for A and so for $|A-i|$, $|A-i|\mathcal{D}_{12}$ is dense in $\mathcal{H}$. Therefore, $Y_1Y_2 = Y_2Y_1$. Hence $Y_1\overline{B_2} = Y_1|A-i|Y_2 = |A-i|Y_2Y_1 = \overline{B_2}Y_1$ on $\mathcal{D}(\overline{B_2})$, i.e., Y_1 commutes with $\overline{B_2}$ and hence with $(\overline{B_2}+i)^{-1}$. Using $\overline{B_1} = |A-i|Y_1$ and the fact that $\overline{B_2}$ and A strongly commute, we therefore obtain

$$\begin{aligned}(\overline{B_2}+i)^{-1}\overline{B_1} &= (\overline{B_2}+i)^{-1}|A-i|Y_1 = |A-i|Y_1(\overline{B_2}+i)^{-1} \\ &= \overline{B_1}(\overline{B_2}+i)^{-1} \quad \text{on } \mathcal{D}(\overline{B_1}).\end{aligned}$$

This implies that $\overline{B_1}$ and $\overline{B_2}$ strongly commute. □

PROPOSITION 3. *Suppose that there are linear subspaces* $\mathcal{D}_0 \subseteq \mathcal{D}(B_1B_2) \cap \mathcal{D}(B_2B_1) \cap \mathcal{D}(A^2)$ *and* $\mathcal{D}_1 \subseteq \mathcal{D}(AB_1) \cap \mathcal{D}(B_1A)$ *such that* $B_1B_2\varphi = B_2B_1\varphi$ *for* $\varphi \in \mathcal{D}_0$ *and* $AB_1\psi = B_1A\psi$ *for* $\psi \in \mathcal{D}_1$. *Suppose that* $\mathcal{D}_0$ *is a core for* A^2 *and that* $\mathcal{D}_1$ *is a core for* A.

Then, $\overline{B_1}$ *is selfadjoint and* $(\overline{B_1}+i)^{-1}$ *commutes with* $\overline{B_2}$. *In particular, if in addition* $\overline{B_2}$ *is selfadjoint, then* $\overline{B_1}$ *and* $\overline{B_2}$ *strongly commute.*

PROOF. From Lemma 1 we conclude that $\overline{B_1}$ is selfadjoint and $\overline{B_1}$ and A strongly commute. By (1), there is a bounded operator X_2 on $\mathcal{H}$ such that $\overline{B_2} = X_2(A+i)$ on $\mathcal{D}(A)$. Note that $\mathcal{D}(A) \subseteq \mathcal{D}(\overline{B_2})$ because of (1).

Suppose $\psi \in \mathcal{D}(A^2)$. Since $\mathcal{D}_0$ is a core for A^2, there is a sequence $(\varphi_n;\ n \in \mathbf{N})$ of vectors from $\mathcal{D}_0$ such that $\varphi = \lim_n \varphi_n$ and $A^2\varphi = \lim_n A^2\varphi_n$ in $\mathcal{H}$. We have for $n \in \mathbf{N}$,

$$\begin{aligned} B_2(B_1+i)\varphi_n &= X_2(A+i)(B_1+i)\varphi_n = (B_1+i)B_2\varphi_n \\ &= (B_1+i)X_2(A+i)\varphi_n. \end{aligned} \tag{4}$$

Since $\mathcal{D}(A^2) \subseteq \mathcal{D}((\overline{B_1}+i)(A+i))$ by (1), $\varphi_n \in \mathcal{D}((\overline{B_1}+i)(A+i))$ for $n \in \mathbf{N}$. Therefore, since $\overline{B_1}$ and A strongly commute, (4) gives

$$X_2(\overline{B_1}+i)(A+i)\varphi_n = (B_1+i)X_2(A+i)\varphi_n \quad \text{for } n \in \mathbf{N}. \tag{5}$$

By (1),

$$\|(\overline{B_1}+i)(A+i)\varsigma\| \le \lambda\|(A+i)^2\varsigma\| + \|(A+i)\varsigma\| \quad \text{for } \varsigma \in \mathcal{D}_0.$$

Thus, letting $n \to \infty$, the limit in (5) exists and yields

$$X_2(\overline{B_1}+i)(A+i)\varphi = (\overline{B_1}+i)X_2(A+i)\varphi. \tag{6}$$

Substituting $\varphi = (A+i)^{-1}\psi$ with $\psi \in \mathcal{D}(A)$, (6) leads to

$$X_2(\overline{B_1}+i)\psi = (\overline{B_1}+i)X_2\psi \quad \text{for all } \psi \in \mathcal{D}(A). \tag{7}$$

Since $\overline{B_1}$ and A strongly commute, $\mathcal{D}(A)$ is a core for $\overline{B_1}$. Hence (7) holds for all $\psi \in \mathcal{D}(B_1)$ and shows that X_2 and $\overline{B_1}$ commute. For $\varphi \in \mathcal{D}(B_2)$, we therefore obtain

$$\begin{aligned} (\overline{B_1}+i)^{-1}B_2\varphi &= (\overline{B_1}+i)^{-1}X_2(A+i)\varphi = X_2(\overline{B_1}+i)^{-1}(A+i)\varphi \\ &= X_2(A+i)(\overline{B_1}+i)^{-1}\varphi = \overline{B_2}(\overline{B_1}+i)^{-1}\varphi, \end{aligned}$$

where we used once more the strong commutativity of $\overline{B_1}$ and A. Hence $(\overline{B_1}+i)^{-1}\overline{B_2} \subseteq \overline{B_2}(\overline{B_1}+i)^{-1}$. □

2. Commutants of unbounded operator algebras. We begin with some terminology. It will be also used in §3.

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. An O^*-*algebra* $\mathcal{A}$ on $\mathcal{D}$ is a $*$-algebra of linear operators defined on $\mathcal{D}$ and leaving $\mathcal{D}$ invariant which contains the identity map I of $\mathcal{D}$. The multiplication in $\mathcal{A}$ is the composition of the operators and the involution in $\mathcal{A}$ is the map $a \to a^+ := a^* \upharpoonright \mathcal{D}$. Suppose that $\mathcal{A}$ is an O^*-algebra on $\mathcal{D}$. Define $\mathcal{A}_I := \{a \in \mathcal{A}\colon \|\varphi\| \le \|a\varphi\| \text{ for all } \varphi \in \mathcal{D}\}$, $\hat{\mathcal{D}}(\mathcal{A}) := \bigcap_{a\in\mathcal{A}} \mathcal{D}(\overline{a})$ and $\mathcal{D}^*(\mathcal{A}) := \bigcap_{a\in\mathcal{A}} \mathcal{D}(a^*)$. The O^*-algebra $\mathcal{A}$ on $\mathcal{D}$ is called *closed* if $\mathcal{D} = \hat{\mathcal{D}}(\mathcal{A})$ and *selfadjoint* if $\mathcal{D} = \mathcal{D}^*(\mathcal{A})$. The *graph topology* of $\mathcal{A}$ is the locally convex topology $\mathrm{t}_{\mathcal{A}}$ on $\mathcal{D}$ which is defined by the seminorms $\|\cdot\|_a := \|a\cdot\|$, $a \in \mathcal{A}$. We denote by

$\mathcal{B}_{\mathcal{A}}(\mathcal{D},\mathcal{D})$ the vector space of all continuous sesquilinear forms (linear in the first and conjugate-linear in the second variable) of $\mathcal{D}|\mathfrak{t}_{\mathcal{A}}| \times \mathcal{D}|\mathfrak{t}_{\mathcal{A}}|$ and by $\mathcal{L}_{\mathcal{A}}(\mathcal{D},\mathcal{H})$ the vector space of all continuous linear mappings of $\mathcal{D}|\mathfrak{t}_{\mathcal{A}}|$ into $\mathcal{H}$. If $\mathfrak{c} \in \mathcal{B}_{\mathcal{A}}(\mathcal{D},\mathcal{D})$, then the sesquilinear form $\mathfrak{c}^+$ defined by $\mathfrak{c}^+(\varphi,\psi) := \overline{\mathfrak{c}(\psi,\varphi)}$, $\varphi,\psi \in \mathcal{D}$, is also in $\mathcal{B}_{\mathcal{A}}(\mathcal{D},\mathcal{D})$.

Let $\mathfrak{A}$ be an abstract $*$-algebra with unit element $\mathbf{1}$. A *$*$-representation* of A on $\mathcal{D}$ is a $*$-homomorphism of $\mathfrak{A}$ onto an O^*-algebra on $\mathcal{D}$ which maps 1 into the identity map on $\mathcal{D}$. A $*$-representation π is called *selfadjoint* if the O^*-algebra $\pi(\mathfrak{A})$ is selfadjoint. As usual, we write $T\eta\mathfrak{N}$ if T is a closed linear operator affiliated with the von Neumann algebra $\mathfrak{N}$.

DEFINITION 4. *Let $\mathcal{A}$ be an O^*-algebra on $\mathcal{D}$. Define*

$$\mathcal{A}_f^c = \{\mathfrak{c} \in \mathcal{B}_{\mathcal{A}}(\mathcal{D},\mathcal{D}) : \mathfrak{c}(a\varphi,\psi) = \mathfrak{c}(\varphi,a^+\psi) \text{ for all } \varphi,\psi \in \mathcal{D} \text{ and } a \in \mathcal{A}\},$$
$$\mathcal{A}_w^c = \{T \in \mathcal{L}_{\mathcal{A}}(\mathcal{D},\mathcal{H}) : \langle Ta\varphi,\psi\rangle = \langle T\varphi,a\psi\rangle \text{ for all } \varphi,\psi \in \mathcal{D} \text{ and } a \in \mathcal{A}\},$$
$$\mathcal{A}_s^c = \{T \in \mathcal{L}_{\mathcal{A}}(\mathcal{D},\mathcal{H}) : T\mathcal{D} \subseteq \mathcal{D} \text{ and } Ta\varphi = aT\varphi \text{ for all } \varphi \in \mathcal{D} \text{ and } a \in \mathcal{A}\},$$
$$\mathcal{A}'_w = \{T \in \mathbf{B}(H) : T \upharpoonright \mathcal{D} \in \mathcal{A}_w^c\} \quad \text{and} \quad \mathcal{A}'_s = \{T \in \mathbf{B}(\mathcal{H}) : T \upharpoonright \mathcal{D} \in \mathcal{A}_s^c\}.$$

We call $\mathcal{A}_f^c$ the form commutant, $\mathcal{A}_w^c$ the weak unbounded commutant, $\mathcal{A}_s^c$ the strong unbounded commutant, $\mathcal{A}'_w$ the weak commutant and $\mathcal{A}'_s$ the strong commutant of the O^*-algebra $\mathcal{A}$.

REMARKS. 1. We identify an operator T on $\mathcal{D}$ with the associated sesquilinear $\mathfrak{c}_T(\varphi,\psi) := \langle T\varphi,\psi\rangle$, $\varphi,\psi \in \mathcal{D}$. Then, obviously, $\mathcal{A}_w^c \subseteq \mathcal{A}_f^c$.

2. The definition of $\mathcal{A}_w^c$ can be slightly reformulated as $\mathcal{A}_w^c = \{T \in \mathcal{L}_{\mathcal{A}}(\mathcal{D},\mathcal{H}) : T\mathcal{D} \subseteq \mathcal{D}^*(\mathcal{A})$ and $Ta\varphi = (a^+)^*T\varphi$ for all $\varphi \in \mathcal{D}$ and $a \in \mathcal{A}\}$. Hence $\mathcal{A}_s^c \subseteq \mathcal{A}_w^c$ and $\mathcal{A}'_s \subseteq \mathcal{A}'_w$. Moreover, from the above characterization we see that if $\mathcal{A}$ is selfadjoint, then $\mathcal{A}_s^c = \mathcal{A}_w^c$ and $\mathcal{A}'_s = \mathcal{A}'_w$.

3. If $\mathcal{A}'_w = \mathcal{A}'_s$, then it follows easily that $\mathcal{A}'_s$ is a von Neumann algebra (see e.g. [**2**]). In this case we write $\mathcal{A}'$ for $\mathcal{A}'_s = \mathcal{A}'_w$.

4. From the definition it is clear that $\mathcal{A}_f^c$ is invariant under the involution $\mathfrak{c} \to \mathfrak{c}^+$. In §3 we show that a similar result for $\mathcal{A}_w^c$ or $\mathcal{A}_s^c$ is not true in general even not if $\mathcal{A}$ is selfadjoint.

5. If $\mathcal{A}_f^c = \mathcal{A}_w^c = \mathcal{A}_s^c$, then $\mathcal{A}^c := \mathcal{A}_s^c$ is an O^*-algebra on $\mathcal{D}$.

PROOF. Suppose $T \in \mathcal{A}^c$. Then, by the invariance of $\mathcal{A}_f^c$ under the involution $\mathfrak{c} \to \mathfrak{c}^+$, $(\mathfrak{c}_T)^+ \in \mathcal{A}_f^c = \mathcal{A}_w^c$. Hence there is a $T_1 \in \mathcal{A}_w^c$ such that $(\mathfrak{c}_T)^+ = \mathfrak{c}_{T_1}$. Obviously, $T_1 = T^* \upharpoonright \mathcal{D}$, so that $T^* \upharpoonright \mathcal{D} \in \mathcal{A}^c$. Since $\mathcal{A}_s^c$ is an algebra of operators leaving $\mathcal{D}$ invariant, this implies that $\mathcal{A}^c$ is an O^*-algebra on $\mathcal{D}$.

In §3 we shall see that the selfadjointness of an O^*-algebra $\mathcal{A}$ is not sufficient to ensure that $\mathcal{A}_f^c = \mathcal{A}_w^c$. A simple sufficient condition (which applies also to some nonselfadjoint O^*-algebras) for the equality of $\mathcal{A}_f^c$ and $\mathcal{A}_w^c$ is given by

PROPOSITION 5. *Let $\mathcal{A}$ be an O^*-algebra on $\mathcal{D}$. Suppose there is an indexed subset $\{a_{\mathfrak{j}}, \mathfrak{j} \in \mathfrak{J}\}$ of operators from $\mathcal{A}_I$ such that $a_{\mathfrak{j}}^2\mathcal{D}$ is dense in H for each $\mathfrak{j} \in \mathfrak{J}$ and such that $\|\cdot\|_{a_{\mathfrak{j}}}$, $\mathfrak{j} \in \mathfrak{J}$, is a directed family of seminorms generating the graph topology $\mathfrak{t}_{\mathcal{A}}$ on $\mathcal{D}$. Then, $\mathcal{A}_f^c = \mathcal{A}_w^c$.*

PROOF. (We generalize an argument used in [**6**].) Since always $\mathcal{A}_w^c \subseteq \mathcal{A}_f^c$, we have to prove that $\mathcal{A}_f^c \subseteq \mathcal{A}_w^c$. For suppose $\mathfrak{c} \in \mathcal{A}_f^c$. Since $\mathfrak{c} \in \mathcal{B}_{\mathcal{A}}(\mathcal{D},\mathcal{D})$, there is an index $\mathrm{j} \in \mathfrak{J}$ and a constant λ such that $|c(\varphi, \psi)| \leq \lambda \|a_{\mathrm{j}}\varphi\| \, \|a_{\mathrm{j}}\psi\|$ for all $\varphi, \psi \in \mathcal{D}$. Since $a_{\mathrm{j}} \in \mathcal{A}_I$, $(\mathcal{D}(\overline{a}_{\mathrm{j}}), \|\cdot\|_{\mathrm{j}} := \|\overline{a}_{\mathrm{j}} \cdot \|)$ is a Hilbert space. Hence there exists a bounded operator X on $\mathcal{H}$ such that $\mathfrak{c}(\varphi, \psi) = \langle X a_{\mathrm{j}}\varphi, a_{\mathrm{j}}\psi\rangle$ for all $\varphi, \psi \in \mathcal{D}$. Since $a_{\mathrm{j}}^{+} \in \mathcal{A}$, we have for $\varphi, \psi \in \mathcal{D}$,

$$\mathfrak{c}(a_{\mathrm{j}}^{+}\varphi, \psi) = \langle X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi, a_{\mathrm{j}}\psi\rangle = \mathfrak{c}(\varphi, a_{\mathrm{j}}\psi) = \langle X a_{\mathrm{j}}\varphi, a_{\mathrm{j}}^{2}\psi\rangle. \tag{8}$$

Define $Y(a_{\mathrm{j}}\varphi) := \varphi$ for $\varphi \in \mathcal{D}$. Since $a_{\mathrm{j}} \in \mathcal{A}_I$, Y is a bounded operator from $a_{\mathrm{j}}\mathcal{D}$ into $\mathcal{H}$. Clearly, Y has an extension to an operator from $\mathbf{B}(\mathcal{H})$ which is again denoted by Y. Letting $\varsigma = a_{\mathrm{j}}^{2}\psi$, (8) implies that $\langle X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi, Y\varsigma\rangle = \langle X a_{\mathrm{j}}\varphi, \varsigma\rangle$ for all $\varsigma \in a_{\mathrm{j}}^{2}\mathcal{D}$. Since $a_{\mathrm{j}}^{2}\mathcal{D}$ is assumed to be dense in $\mathcal{H}$, $Y^{*} X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi = X a_{\mathrm{j}}\varphi$ for $\varphi \in \mathcal{D}$. Thus

$$\mathfrak{c}(\varphi, \psi) = \langle Y^{*} X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi, a_{\mathrm{j}}\psi\rangle = \langle X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi, Y a_{\mathrm{j}}\psi\rangle = \langle X a_{\mathrm{j}} a_{\mathrm{j}}^{+}\varphi, \psi\rangle$$

for $\varphi, \psi \in \mathcal{D}$, that is, $\mathfrak{c} = \mathfrak{c}_{X a_{\mathrm{j}} a_{\mathrm{j}}^{+}}$. Obviously, $a_{\mathrm{j}} a_{\mathrm{j}}^{+} \in \mathcal{L}_{\mathcal{A}}(\mathcal{D}, \mathcal{H})$. Since $\mathfrak{c} \in \mathcal{A}_{f}^{c}$, $X a_{\mathrm{j}} a_{\mathrm{j}}^{+} \in \mathcal{A}_{w}^{c}$. □

PROPOSITION 6. *Suppose $\mathcal{A}$ is a closed O^*-algebra on $\mathcal{D}$ and $\mathfrak{N}$ is a von Neumann algebra contained in the strong commutant $\mathcal{A}_s'$. Let T be a closed linear operator on $\mathcal{H}$ which is affiliated with $\mathfrak{N}$. If $\mathcal{D} \subseteq \mathcal{D}(T)$ and $T \restriction \mathcal{D} \in \mathcal{L}_{\mathcal{A}}(\mathcal{D}, \mathcal{H})$, then $T \restriction \mathcal{D} \in \mathcal{A}_s^c$.*

PROOF. Let $T = U|T|$ be the polar decomposition of T and let $|T| = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral decomposition of the positive selfadjoint operator $|T|$. Since $T\eta\mathfrak{N}$, $U \in \mathfrak{N}$ and $|T|\eta\mathfrak{N}$. Therefore, $|T|E(0,n) \in \mathfrak{N}$ and hence $T_n := TE(0,n) = U|T|E(0,n) \in \mathfrak{N}$ for $n \in \mathbf{N}$. Since $\mathfrak{N} \subseteq \mathcal{A}_s'$, we have $T_n\mathcal{D} \subseteq \mathcal{D}$ and $T_n a\varphi = aT_n\varphi$ for $a \in \mathcal{A}$, $\varphi \in \mathcal{D}$ and $n \in \mathbf{N}$. Suppose $\varphi \in \mathcal{D}$ and $a \in \mathcal{A}$. From $T\varphi = U|T|\varphi = \lim_n U|T|E(0,n)\varphi = \lim_n T_n\varphi$ and $Ta\varphi = \lim T_n a\varphi$ we therefore conclude that $T\varphi \in \mathcal{D}(\overline{a})$ and $\overline{a}T\varphi = Ta\varphi$. Since $\mathcal{A}$ is a closed O^*-algebra on $\mathcal{D}$, $\mathcal{D} = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\overline{a})$. Thus $T\varphi \in \mathcal{D}$ and $aT\varphi = Ta\varphi$ for all $\varphi \in \mathcal{D}$ and $a \in \mathcal{A}$ which gives the assertion. □

PROPOSITION 7. *Let $\mathcal{A}$ be an O^*-algebra on $\mathcal{D}$ and let $\mathfrak{N}$ be a von Neumann algebra contained in the weak commutant $\mathcal{A}_w'$. Suppose T is a closed linear operator on H such that $T\eta\mathbf{N}$, $\mathcal{D} \subseteq \mathcal{D}(T)$ and $T \restriction \mathcal{D} \in \mathcal{L}_{\mathcal{A}}(\mathcal{D}, \mathcal{H})$. Then, $T \restriction \mathcal{D} \in \mathcal{A}_w^c$.*

PROOF. Up to the following slight modification, the proof follows the lines of the preceding proof. Since $T_n \in \mathfrak{N} \subseteq \mathcal{A}_w'$, we have $T_n a\varphi = a^* T_n\varphi$ for $a \in \mathcal{A}$, $\varphi \in \mathcal{D}$ and $n \in \mathbf{N}$. Since $\mathcal{D}^*(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} \mathcal{D}(a^*)$ by definition, it follows that $T\mathcal{D} \subseteq \mathcal{D}^*(\mathcal{A})$ and similarly as above $T \restriction \mathcal{D} \in \mathcal{A}_w'$. □

COROLLARY 8. *Suppose $\mathcal{A}$ is a selfadjoint O^*-algebra on $\mathcal{D}$ and T is a closed linear operator on $\mathcal{H}$ which is affiliated with the von Neumann algebra $\mathcal{A}'$ ($\equiv \mathcal{A}_s' = \mathcal{A}_w'$) such that $\mathcal{D} \subseteq \mathcal{D}(T)$ and $T \restriction \mathcal{D} \in \mathcal{L}_{\mathcal{A}}(\mathcal{D}, \mathcal{H})$. Then $T \restriction \mathcal{D} \in \mathcal{A}_s^c \equiv \mathcal{A}_w^c$.*

The converse of Corollary 8 is not true in general. To be more precise, if T is a symmetric linear operator contained in the strong (or weak) unbounded commutant of a selfadjoint O^*-algebra $\mathcal{A}$, then T is not necessarily affiliated with the von Neumann algebra $\mathcal{A}'$. (A counterexample is described in Remark 2, in §3.) Concerning the latter assertion, the following condition for an O^*-algebra $\mathcal{A}$ on $\mathcal{D}$ is useful:

(I) *There are an indexed set $\{a_j; j \in \mathfrak{J}\}$ of symmetric operators from $\mathcal{A}$ and an indexed set $\{\alpha_j; j \in \mathfrak{J}\}$ of complex numbers such that a_j^2 is essentially selfadjoint on $\mathcal{D}$ for each $j \in \mathfrak{J}$ and such that $\|\cdot\|_{a_j+\alpha_j}$, $j \in \mathfrak{J}$, is a directed family of seminorms generating the graph topology $\mathfrak{t}_{\mathcal{A}}$ on $\mathcal{D}$.*

Conditions of a similar kind have been invented and used by Araki and Jurzak in [**1**]. The above condition (I) is a (slight) modification of the condition (I_0') formulated in [**1**, p. 1015].

LEMMA 9. *Let $\mathcal{A}$ be a closed O^*-algebra on $\mathcal{D}$ satisfying* (I). *Then, for each $j \in \mathfrak{J}$, $\overline{a}_j$ is a selfadjoint operator. The O^*-algebra $\mathcal{A}$ is then selfadjoint, $\mathcal{A}_f^c = \mathcal{A}_w^c = \mathcal{A}_s^c$ and $\mathcal{A}_w' = \mathcal{A}_s'$. Moreover, $\mathcal{A}^c := \mathcal{A}_s^c$ is an O^*-algebra on $\mathcal{D}$ and $\mathcal{A}' := \mathcal{A}_s'$ is a von Neumann algebra.*

PROOF. Let $j \in \mathfrak{J}$. Since $a_j\mathcal{D} \subseteq \mathcal{D}$, $\ker(a_j^* + i) \subseteq \ker((a_j^2)^* + I)$. Since a_j^2 is essentially selfadjoint by (I), the latter is trivial, so that $\overline{a}_j$ is selfadjoint.

The assumptions concerning the seminorms $\|\cdot\|_{a_j+\alpha_j}$, $j \in \mathfrak{J}$, imply that $\hat{\mathcal{D}}(\mathcal{A}) = \bigcap_{j\in\mathfrak{J}} \mathcal{D}(\overline{a_j + \alpha_j})$. Therefore, since $\mathcal{A}$ is closed, $\mathcal{D} = \bigcap_{j\in\mathfrak{J}} \mathcal{D}(\overline{a}_j) = \bigcap_{j\in\mathfrak{J}} \mathcal{D}(a_j^*)$. Hence $\mathcal{D} = \mathcal{D}^*(\mathcal{A})$, so that $\mathcal{A}$ is selfadjoint.

We prove that $\mathcal{A}_f^c = \mathcal{A}_w^c$. Without loss of generality we can assume that $a_j + \alpha_j \in \mathcal{A}_I$ for each $j \in \mathfrak{J}$. Then $-\alpha_j$ belongs to the resolvent set $\rho(\overline{a_j})$ of the selfadjoint operator $\overline{a_j}$. Hence $-|\alpha_j|^2 \in \rho((\overline{a_j})^2)$ and $X_j := (\overline{a_j} + \alpha_j)^2((\overline{a_j})^2 + |\alpha_j|^2)^{-1}$ is an isomorphism of the Hilbert space $\mathcal{H}$. Since a_j^2 is essentially selfadjoint, $(a_j^2 + \alpha_j^2)\mathcal{D}$ and so $(a_j + \alpha_j)^2\mathcal{D} = X_j(a_j^2 + |\alpha_j|^2)\mathcal{D}$ is dense in $\mathcal{H}$. Thus the assumptions of Proposition 5 (with a_j replaced by $a_j + \alpha_j$) are satisfied and we conclude that $\mathcal{A}_f^c = \mathcal{A}_w^c$.

The other assertions follow from the remarks after Definition 4. □

Under more restrictive assumptions the assertion of the following proposition has been stated in [**1**, Theorem 3].

PROPOSITION 10. *If T is a symmetric operator from $\mathcal{A}^c$, then T is selfadjoint and affiliated with the von Neumann algebra $\mathcal{A}'$.*

PROOF. Suppose $a \in \mathcal{A}$. Since $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{D}, \mathcal{H})$ and $\mathfrak{t}_{\mathcal{A}}$ is generated by the directed family of seminorms $\|\cdot\|_{a_j+\alpha_j}$, $j \in \mathfrak{J}$, there is $j \in \mathfrak{J}$ and a constant λ such that $\|a\varphi\| \le \lambda\|(a_j + \alpha_j)\varphi\|$ and $\|T\varphi\| \le \lambda\|(a_j + \alpha_j)\varphi\|$ for $\varphi \in \mathcal{D}$. Moreover, by Lemma 9, $\overline{a_j}$ is selfadjoint. Therefore, Proposition 3 applies (in case $B_1 = a$, $B_2 = T$, $A = \overline{a_j}$, $\mathcal{D}_0 = \mathcal{D}_1 = \mathcal{D}$) and shows that T is selfadjoint and that $(\overline{T} + i)^{-1}$ and $\overline{a}$ commute. Hence $(\overline{T}+i)^{-1}\mathcal{D} \subseteq \mathcal{D}(\overline{a})$ for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is closed, $\mathcal{D} = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\overline{a})$, so $(\overline{T} + i)^{-1}\mathcal{D} \subseteq \mathcal{D}$. Therefore, $(\overline{T} + i)^{-1} \in \mathcal{A}'$ which implies $T\eta\mathcal{A}'$. □

3. A counterexample. In this section let $\mathfrak{A}$ denote the $*$-algebra with unit of all complex polynomials in two commuting hermitian indeterminants x_1 and x_2. Our main result in this section is

PROPOSITION 11. *There exists a selfadjoint $*$-representation π of $\mathfrak{A}$ and an operator $T \in \pi(\mathfrak{A})_w^c$ such that $\mathcal{D}$ is not contained in $\mathcal{D}(T^*)$. In particular, $\pi(\mathfrak{A})_f^c \neq \pi(\mathfrak{A})_w^c$.*

Before proving Proposition 11, we recall some notation and some facts mainly taken from [**8** and **9**]. Let A and B be selfadjoint operators on a Hilbert space $\mathcal{H}$ with bounded inverses $X := A^{-1}$ and $Y := B^{-1}$. Suppose $n, m \in \mathbf{N}$. Let Q_{nm} denote the projection of $\mathcal{H}$ onto the closed linear span of $[X^k, Y^l]\mathcal{H}$, where $k = 1, \ldots, n$ and $l = 1, \ldots, m$, and let $\mathcal{D}_{nm} := X^n Y^m (I - Q_{nm})\mathcal{H}$. By Lemma 1.2 in [**8**], we have

(a) $X^k Y^l \varphi = Y^l X^k \varphi$ for $\varphi \in (I - Q_{nm})\mathcal{H}$, $k, l, n, m \in \mathbf{N}$, $k \leq n$, $l \leq m$.

Put $Q_{k0} = Q_{0k} = 0$ for $k = 0, 1, \ldots$. Consider the following conditions:

$(\mathrm{I})_{\mathrm{nm}}$ if $X\varphi \in Q_{nm}\mathcal{H}$ for some $\varphi \in \mathcal{H}$, then $\varphi \in Q_{n-1,m}\mathcal{H}$.

$(\mathrm{II})_{\mathrm{nm}}$ If $Y\varphi \in Q_{nm}\mathcal{H}$ for some $\varphi \in \mathcal{H}$, then $\varphi \in Q_{n,m-1}\mathcal{H}$.

Assume that $(\mathrm{I})_{\mathrm{nm}}$ and $(\mathrm{II})_{\mathrm{nm}}$ hold for all $n, m \in \mathbf{N}$. Then

(b) *$\mathcal{D}_\infty := \bigcap_{n,m\in\mathbf{N}} \mathcal{D}_{nm}$ is a core for both A and B. There is a (unique) selfadjoint $*$-representation π of $\mathfrak{A}$ on $\mathcal{D}_\infty$ such that $\pi(x_1) = A \restriction \mathcal{D}_\infty$ and $\pi(x_2) = B \restriction \mathcal{D}_\infty$.*

(c) *For arbitrary $r, s \in \mathbf{N}$, $\overline{A^r B^s \restriction \mathcal{D}_\infty} = \overline{A^r B^s \restriction \mathcal{D}_{rs}}$.*

Indeed, (b) follows immediately by combining the results in [**8**, §1], with Proposition 3.3 in [**9**]. Though (c) is not explicitly stated in [**8**], it follows by arguing as in the proof of Proposition 1.5 in [**8**]. Let $\|\cdot\|_{rs}$ denote the norm on $\mathcal{D}_{rs}$ defined by $\|\cdot\|_{rs} := \|B^s A^r \cdot\|$. Then the argument of the proof of Proposition 1.5 shows that $(\mathrm{I})_{\mathrm{nm}}$ and $(\mathrm{II})_{\mathrm{nm}}$ for all $n, m \in \mathbf{N}$ imply that $(\mathcal{D}_\infty, \|\cdot\|_{rs})$ is dense in $(\mathcal{D}_{rs}, \|\cdot\|_{rs})$. But this is (c).

PROOF OF PROPOSITION 11. We now specialize the operators A and B from above. Let S be the unilateral shift on the Hardy space $\mathcal{H} := \mathcal{H}^2(\mathbf{T})$, i.e., $(S\varphi)(z) = z\varphi(z)$ for $\varphi \in \mathcal{H}$. Let $X := \operatorname{Re} S$, $Y := \operatorname{Im} S$, $A := X^{-1}$ and $B := Y^{-1}$. It is then straightforward to check that $Q_{nm}\mathcal{H} = \operatorname{Lin}\{z^0, z, \ldots, z^{n+m-2}\}$ and that $(\mathrm{I})_{\mathrm{nm}}$ and $(\mathrm{II})_{\mathrm{nm}}$ are fulfilled for arbitrary $n, m \in \mathbf{N}$. Let π be the corresponding selfadjoint $*$-representation of $\mathfrak{A}$ on $\mathcal{D}$ which exists by (b). Define $T := S^{*2}AB \restriction \mathcal{D}_\infty$.

First we prove that $T \in \pi(\mathfrak{A})^c_w$. Obviously, $T \in \mathcal{L}_{\pi(\mathfrak{A})}(\mathcal{D}_\infty, \mathcal{H})$. To prove that $T \in \pi(\mathfrak{A})^c_w$, it suffices to show that for $l = 1, 2$ and $\varphi, \psi \in \mathcal{D}_\infty$

$$\langle T\pi(x_l)\varphi, \psi\rangle = \langle T\varphi, \pi(x_l)\psi\rangle. \tag{9}$$

For suppose $\varphi, \psi \in \mathcal{D}_\infty$. Since $\mathcal{D}_\infty \subseteq \mathcal{D}_{21}$, there is $\varphi \in (I - Q_{21})\mathcal{H}$ such that $\varphi = X^2 Y\varsigma$. Further, $\psi = X\xi$ for some $\xi \in \mathcal{H}$. By (a), $\varphi = XYX\varsigma = YX^2\varsigma$. Since $\varsigma \perp z^0$ and $\varsigma \perp z$, $S^{*2}X\varsigma = XS^{*2}\varsigma$. Therefore,

$$\begin{aligned}\langle T\pi(X_1)\varphi, \psi\rangle &= \langle S^{*2}ABA\varphi, \psi\rangle = \langle S^{*2}ABAXYX\varsigma, X\xi\rangle = \langle XS^{*2}\varsigma, \xi\rangle \\ &= \langle S^{*2}X\varsigma, \xi\rangle = \langle S^{*2}ABYX^2\varsigma, \xi\rangle = \langle S^{*2}AB\varphi, A\psi\rangle = \langle T\varphi, \pi(x_1)\psi\rangle.\end{aligned}$$

A similar reasoning shows (9) in case $l = 2$.

The proof of $\mathcal{D}_\infty \nsubseteq \mathcal{D}(T^*)$ will be indirect. Assume that $\mathcal{D}_\infty \subseteq \mathcal{D}(T^*)$. Then $R := T^* \restriction \mathcal{D}_\infty$ is a closable operator on $\mathcal{D}$. Because $\mathcal{D}_\infty[\mathfrak{t}_{\pi(\mathfrak{A})}]$ is a Fréchet space, the closed graph theorem ensures that R maps $\mathcal{D}_\infty[\mathfrak{t}_{\pi(\mathfrak{A})}]$ continuously into $\mathcal{H}$. Since $\mathfrak{t}_{\pi(\mathfrak{A})}$ is generated by the directed family of norms$\|A^n B^n \cdot\|$, $n \in \mathbf{N}$, there is a bounded operator Z on $\mathcal{H}$ and $n \in \mathbf{N}$ such that $R = ZA^n B^n \restriction \mathcal{D}_\infty$. Then we have

$$\langle S^{*2}AB\varphi, \psi\rangle = \langle \varphi, ZA^n B^n \psi\rangle \quad \text{for all } \varphi, \psi \in \mathcal{D}_\infty. \tag{10}$$

From (c) it follows that (10) is valid for arbitrary $\varphi \in \mathcal{D}_{11}$ and $\psi \in \mathcal{D}_{nn}$. Since $\mathcal{D}_{11} = YX(I - Q_{11})\mathcal{H}$ and $\mathcal{D}_{nn} = X^n Y^n (I - Q_{nn})\mathcal{H}$, it therefore follows from (10) that

$\langle S^{*2}(I-Q_{11})\varsigma, X^nY^n(I-Q_{nn})\xi\rangle = \langle YX(I-Q_{11})\varsigma, Z(I-Q_{nn})\xi\rangle$ for all $\varsigma, \xi \in \mathcal{H}$,

i.e., $(I-Q_{11})(S^2X^nY^n - XYZ)(I-Q_{nn}) = 0$. Consequently,

$$S^2X^nY^n(I-Q_{nn})\mathcal{H} \subseteq XY\mathcal{H} + Q_{11}\mathcal{H}. \tag{11}$$

For $k, l = 1, \ldots, n$, X^k and Y^l commute on $(I-Q_{nn})\mathcal{H}$ by (a), hence so do S^k and S^{*l}. Thus (11) leads to

$$S^2(S+S^*)(S-S^*)(S^2-S^{*2})^{n-1}(I-Q_{nn})\mathcal{H} \subseteq (S+S^*)(S-S^*)\mathcal{H} + Q_{11}\mathcal{H}. \tag{12}$$

Using that $(S+S^*)\mathcal{H} \cap Q_{11}\mathcal{H} = \{0\}$ (which is true by $(\mathrm{I})_{11}$), $\ker(S+S^*) = \{0\}$ and the identity

$$S^2(S+S^*)(S-S^*) = (S+S^*)(S-S^*)(S^2+Q_{11}) + (S+S^*)Q_{11}S^*,$$

it follows from (12) that

$$Q_{11}S^*(S^2-S^{*2})^{n-1}(I-Q_{nn})\mathcal{H} \subseteq (S-S^*)\mathcal{H} \equiv Y\mathcal{H}.$$

But $Q_{11}S^*(S^2-S^{*2})^{n-1}(I-Q_{nn})z^{2n-1} = (-1)^{n-1}z \notin Y\mathcal{H}$ by $(\mathrm{I})_{11}$ which is the desired contradiction. That is, we have proven that $\mathcal{D}_\infty \not\subseteq \mathcal{D}(T^*)$.

Finally we check that $\pi(\mathfrak{A})^c_f \neq \pi(\mathfrak{A})^c_w$. Since $T \in \pi(\mathfrak{A})^c_w$, $\mathfrak{c}_T$ and so $(\mathfrak{c}_T)^+$ is in $\pi(\mathfrak{A})^c_f$. If there would be a $T_1 \in \pi(\mathfrak{A})^c_w$ such that $(\mathfrak{c}_T)^+ = \mathfrak{c}_{T_1}$, then obviously $T_1 \subseteq T^*$ and so $\mathcal{D}_\infty \subseteq \mathcal{D}(T^*)$ which is the contradiction. Consequently, $\pi(\mathfrak{A})^c_f \neq \pi(\mathfrak{A})^c_w$. $\square$

REMARKS. 1. The operator T from the preceding proof is closable.

2. Let π be the selfadjoint $*$-representation of $\mathfrak{A}$ as constructed in the above proof. Since the shift operator S is irreducible and $\overline{\pi(x_1)} = A$, $\overline{\pi(x_2)} = B$ by (a), it follows easily that the von Neumann algebra $\pi(\mathfrak{A})'$ reduces to the scalars. Since the $*$-algebra A is commutative, $\pi(x_1)$ belongs to $\pi(\mathfrak{A})^c_s$, but $\overline{\pi(x_1)} = A$ is certainly not affiliated with $\pi(\mathfrak{A})'$.

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