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A CONSTRUCTION OF FINITE AND σ-FINITE INVARIANT MEASURES IN MEASURE SPACES

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ABSTRACT. Let T be a bijective nonsingular transformation on a finite measure space. We shall first construct a σ-finite and finite invariant measure by a unified method which is valid for both cases. Secondly we shall give another construction of a finite invariant measure. We shall also give a new necessary and sufficient condition of a unified form for the existence of σ-finite and finite invariant measures. Further, we shall discuss in detail ergodic transformations.

1. Introduction. Let T be a bijective nonsingular transformation on a finite measure space $(X, \mathcal{F}, m)$. This means that T is a bimeasurable bijection and $m(T^{-1}E) = m(TE) = 0$ for any measurable set E with $m(E) = 0$. Let μ and ν be σ-finite measures on $(X, \mathcal{F})$. μ is said to be absolutely continuous with respect to ν if $\mu(E) = 0$ for any measurable set E with $\nu(E) = 0$. In this case we write simply $\mu \ll \nu$. If $\mu \ll \nu$ and $\nu \ll \mu$, then we say that μ and ν are equivalent or μ is equivalent to ν and write simply $\mu \sim \nu$. A measure μ is said to be invariant if $\mu(T^{-1}E) = \mu(TE) = \mu(E)$ for any measurable set E. A measurable set E is invariant if $TE = E$. A transformation T is ergodic if $m(E) = 0$ or $m(E^c) = 0$ for any invariant measurable set E. Construction of finite invariant measures were studied by Hopf [**5**], Dowker [**2**], Calderón [**1**] and Hajian and Kakutani [**3**], while Halmos [**4**] studied σ-finite invariant measures. We shall present a unified construction which is useful for both cases at the same time. Although the idea can be found already in Hopf [**5**] and Halmos [**4**] we shall use general measure theory and give a more systematic treatment. We put $F^* = \bigcup_{n=-\infty}^{\infty} T^n F$ for any measurable set F. We fix a measurable set E. Putting $A' = A \cap E^*$ for any measurable set A, we define m_E by

$$m_E(A) = \inf\left\{ \sum_{n=-\infty}^{\infty} m(T^n A_n), A_n \in \mathcal{F}\ (n = 0, \pm 1, \pm 2, \dots),\right.$$
$$\left. A' \subset \bigcup_{n=-\infty}^{\infty} A_n \text{ and } \bigcup_{n=-\infty}^{\infty} T^n A_n \subset E \right\}.$$

Then m_E is a σ-finite invariant measure with $m_E \ll m$ (Theorem 1). For any σ-finite invariant measure μ with $\mu \ll m$ there exists a measurable set E with $\mu \sim m_E$ (Theorem 2). Further, if μ is finite we can find a measurable set E with $m_E(X) < \infty$. From this we can give a necessary and sufficient condition

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for the existence of a σ-finite or finite invariant measure which is equivalent to m (Corollary 1 or 2). In Theorem 4 we shall give a necessary and sufficient condition for the existence of a σ-finite invariant measure which is equivalent to the given measure m in case T is ergodic. We shall discuss ergodic transformations in detail in Theorems 5 and 6. Finally we shall generalize the well-known fact that an invariant measure exists uniquely in case T is ergodic. Using this idea we shall give another construction of a finite invariant measure.

2. Construction and existence theorems. We first give a unified construction of invariant measures applicable to finite as well as σ-finite cases. We also give a criterion for the existence of an invariant measure which is equivalent to m.

THEOREM 1. *For any measurable set E, m_E is a σ-finite invariant measure with $m_E \ll m$.*

PROOF. We define a set function μ_E by

$$\mu_E(A) = \inf\left\{ \sum_{n=-\infty}^{\infty} m(T^n A_n) : A' \subset \bigcup_{n=-\infty}^{\infty} A_n,\ T^n A_n \subset E \text{ and } A_n \in \mathcal{F}\ (n = 0, \pm 1, \pm 2, \dots) \right\}$$

for any subset A of X. Then μ_E is an outer measure which satisfies $\mu_E(TA) = \mu_E(A)$ for any subset A of X. It is easy to check that any $\mathcal{F}$-measurable set is μ_E-measurable. Therefore we can define a measure on $(X, \mathcal{F})$ by

$$m_E(A) = \inf\left\{ \sum_{n=-\infty}^{\infty} m(T^n A_n) : A' = \bigcup_{n=-\infty}^{\infty} A_n, T^n A_n \subset E \text{ and } A_n \in \mathcal{F}\ (n = 0, \pm 1, \pm 2, \dots) \right\}$$

for any $A \in \mathcal{F}$. The nonsingularity of T implies $m_E \ll m$. m_E is σ-finite since m_E is invariant, $m_E(E) < \infty$ and $m_E(E^{*c}) = 0$.

THEOREM 2. *Let μ be a σ-finite invariant measure with $\mu \ll m$. Then there exists a measurable set E with $\mu \sim m_E$.*

COROLLARY 1. *There exists a σ-finite invariant measure μ with $\mu \sim m$ if and only if there exists a measurable set E with $m_E \gg m$.*

PROOF OF THEOREM 2. Let $\alpha = \sup\{m(A) : \mu(A) = 0\}$. Then there exists an invariant measurable set A such that $m(A) = \alpha$ and $\mu(A) = 0$. Restricting the measures μ and m to the invariant measurable set A^c we have $\mu|_{A^c} \sim m|_{A^c}$. Therefore we need only to prove that there exists a measurable set E with $m_E \sim m$ if there exists a σ-finite invariant measure μ with $\mu \sim m$. By Theorem 1 $m_E \ll m$ for any measurable set E. We now show that there exists a measurable set E with $m_E \gg m$. For the proof we need the following.

LEMMA. *Let μ be a σ-finite invariant measure with $m \ll \mu$ and f a Radon-Nikodým derivative $dm/d\mu$. We put*

$$E = \{x : 0 < a \le f(x) \le b < \infty\}.$$

Then for any measurable set A we have

$$a\mu(A \cap E^*) \le m_E(A) \le b\mu(A \cap E^*).$$

PROOF. By assumption for any measurable set F we have $m(F) = \int_F f\,d\mu$ and

$$m_E(A) = \inf\left\{ \sum_{n=-\infty}^{\infty} m(T^n A_n) : A' \subset \bigcup_{n=-\infty}^{\infty} A_n,\ T^n A_n \subset E \right.$$
$$\left. \text{and } A_n \in \mathcal{F}\ (n = 0, \pm 1, \pm 2, \dots) \right\}.$$

Since A' is also measurable, we may assume that the infimum runs over A_n's with $A' = \bigcup_{n=-\infty}^{\infty} A_n$ (disjoint). We have

$$\mu(A') = \sum_{n=-\infty}^{\infty} \mu(A_n) \quad \text{and} \quad m(T^n A_n) = \int_{T^n A_n} f\,d\mu \qquad (n = 0, \pm 1, \pm 2, \dots).$$

By assumption

$$a\mu(T^n A_n) \le \int_{T^n A_n} f\,d\mu \le b\mu(T^n A_n).$$

Since μ is invariant, we have

$$a\mu(A_n) \le m(T^n A_n) \le b\mu(A_n).$$

We get the lemma by taking the infimum of the summation over n.

We now go back to the proof of Theorem 2 using the same notation as in the Lemma. $f(x) > 0$ a.e., since $\mu \sim m$. We put

$$X_n = \{x : 1/(n+1) \le f(x) \le n+1\} \quad \text{for } n = 1, 2, \dots,$$
$$Y_1 = X_1 \quad \text{and} \quad Y_n = X_n - \sum_{i=1}^{n-1} Y_i^* \quad \text{for } n \ge 2.$$

Then $Y_i^* \cap Y_j^* = \varnothing$ for $i \ne j$ and $m(X - \bigcup_{n=1}^{\infty} Y_n^*) = 0$. Putting $E = \bigcup_{n=1}^{\infty} Y_n$ we obtain a measure m_E with $m \ll m_E$. Indeed if $m_E(A) = 0$, then $m_E(A \cap Y_n^*) = 0$ for any positive integer n. On the other hand, the definition of m_E and Y_n, and $E \cap Y_n^* = Y_n$ imply

$$m_E(A \cap Y_n^*) = m_{Y_n}(A \cap Y_n^*).$$

By the Lemma we have

$$(1/(n+1))\mu(A \cap Y_n^*) \le m_E(A \cap Y_n^*) \le (n+1)\mu(A \cap Y_n^*).$$

This implies $\mu(A \cap Y_n^*) = 0$ and therefore $m(A \cap Y_n^*) = 0$ for $n = 1, 2, 3, \dots$. Thus we obtain $m(A) = 0$.

THEOREM 3. *Let μ be a finite invariant measure with $\mu \ll m$. Then there exists a measurable set E such that $\mu \sim m_E$ and $m_E(X) < \infty$.*

COROLLARY 2. *There exists a finite invariant measure μ with $\mu \sim m$ if and only if there exists a measurable set E such that $m \ll m_E$ and $m_E(X) < \infty$. In this case for any $\varepsilon > 0$ we can find a measurable set E with $m_E(X) < m(X) + \varepsilon$.*

PROOF OF THEOREM 3. As in the case of Theorem 2, the proof of Theorem 3 is reduced to the case where there exists a finite invariant measure μ with $\mu \sim m$.

We need only to prove that there exists a measurable set E with $m_E \gg m$ and $m_E(X) < \infty$. From $\mu \gg m$ and the Radon-Nikodým theorem

$$m(A) = \int_A f\, d\mu \qquad (A \in \mathcal{F}),$$

where $f(x) > 0$ a.e. Let $\varepsilon > 0$ be any real number and choose $\delta > 0$ in such a way that

$$\delta \cdot \max\left\{\mu(X), \int f(x)\, d\mu\right\} < \varepsilon/2. \tag{1}$$

We put $a = 1 + \delta$ and let

$$E_0 = \{x : \delta a^{-1} \le f(x) < \delta\}.$$

We assume that E_i $(n < i \le 0)$ are chosen so that

$$E_i \subset \{x : \delta a^{i-1} \le f(x) < \delta a^i\} \quad \text{for } n < i \le 0$$

and

$$E_i^* \cap E_j^* = \emptyset \quad \text{for } n < i < j \le 0.$$

We put

$$E_n = \{x : \delta a^{n-1} \le f(x) < \delta a^n\} - \bigcup_{i=0}^{n-1} E_i^*.$$

The sequence $\{E_n\}$ is inductively defined for all nonpositive integers. Further, we put $E_- = \bigcup_{n=0}^{\infty} E_{-n}^*$ and

$$E_1 = \{x : \delta \le f(x) < \delta a\} - E_-.$$

We assume that E_i $(1 \le i < n)$ are chosen so that

$$E_i \subset \{x : \delta a^{i-1} \le f(x) < \delta a^i\}$$

and

$$E_i^* \cap E_j^* = \emptyset \quad \text{for } i < j < n.$$

We put

$$E_n = \{x : \delta a^{n-1} \le f(x) < \delta a^n\} - E_- - \bigcup_{i=1}^{n-1} E_i^*.$$

Then $m(\{x : f(x) > 0\} - \bigcup_{n=-\infty}^{\infty} E_n^*) = 0$ and $E_k^* \cap E_l^* = \emptyset$ for $k \ne l$. By the Lemma we have

$$\delta a^{n-1}\mu(E_n^*) \le m_{E_n}(E_n^*) \le \delta a^n \mu(E_n^*) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{2}$$

On the other hand, by (1) we have

$$\sum_{n=-\infty}^{0} \delta a^n \mu(E_n^*) \le \delta\mu(X) < \varepsilon/2. \tag{3}$$

Since $f(x) \ge \delta a^{n-1}$ on E_n^* $(n \ge 1)$ by the definition, we have

$$\int f\, d\mu \ge \sum_{n=1}^{\infty} \int_{E_n^*} f\, d\mu \ge \sum_{n=1}^{\infty} \delta a^{n-1} \mu(E_n^*). \tag{4}$$

Therefore we have

$$\sum_{n=-\infty}^{\infty} m_{E_n}(X) = \sum_{n=-\infty}^{\infty} m_{E_n}(E_n^*) = \sum_{n=-\infty}^{0} m_{E_n}(E_n^*) + \sum_{n=1}^{\infty} m_{E_n}(E_n^*)$$
$$< \varepsilon/2 + a\sum_{n=1}^{\infty} \delta a^{n-1}\mu(E_n^*) \quad \text{(by (2) and (3))},$$
$$< \varepsilon/2 + a\int f\,d\mu < \int f\,d\mu + \varepsilon \quad \text{(by (4) and (1))}.$$

Consequently

$$\sum_{n=-\infty}^{\infty} m_{E_n}(X) < \int f\,d\mu + \varepsilon. \tag{5}$$

Putting $E = \bigcup_{n=-\infty}^{\infty} E_n$, m_E is a σ-finite invariant measure by Theorem 1. The definition of m_E and E_n, and $E_n^* \cap E = E_n$ imply

$$m_E(E_n^*) = m_{E_n}(E_n^*) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

From this and (5) follows

$$m_E(X) = \sum_{n=-\infty}^{\infty} m_E(E_n^*) = \sum_{n=-\infty}^{\infty} m_{E_n}(E_n^*) = \sum_{n=-\infty}^{\infty} m_{E_n}(X) < \int f\,d\mu + \varepsilon.$$

Therefore m_E is a finite invariant measure. We now show that $m_E \gg m$. If $m_E(A) = 0$, then $m_E(A \cap E_n^*) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$. The definition of m_E and E_n, and $E_n^* \cap E = E_n$ imply $m_E(A \cap E_n^*) = m_{E_n}(A \cap E_n^*)$. By the Lemma, we have

$$\delta a^{n-1}\mu(A \cap E_n^*) \le m_E(A \cap E_n^*) \le \delta a^n \mu(A \cap E_n^*),$$

which implies $\mu(A \cap E_n^*) = 0$. From $\mu \gg m$ follows $m(A \cap E_n^*) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$, hence $m(A) = 0$.

Thus by Theorem 1 we obtain Theorem 3.

3. Ergodic transformations. In this section we assume that T is ergodic.

THEOREM 4. *There exists a σ-finite invariant measure which is equivalent to m if and only if there exists a measurable set E for which m_E is nontrivial.*

REMARK. The condition is equivalent to a condition that there exists a measurable set E with $m_E(E) > 0$ (see Corollary 3).

COROLLARY 3. *m_E is a σ-finite invariant measure which is equivalent to m if and only if $m_E(E) > 0$.*

PROOF OF THEOREM 4. By Theorem 2 the condition is necessary. We prove that the condition is sufficient. Suppose that there exists no measurable set E such that $m_E \gg m$. That is, there exists a measurable set A with $m(A) > 0$ and $m_E(A) = 0$. From this follows $m_E(A^*) = 0$ and $m(A^*) > 0$. We have $m(A^{*c}) = 0$ because T is ergodic, which implies $m_E(A^{*c}) = 0$. Hence $m_E(X) = 0$, which contradicts the existence of a nontrivial m_E.

THEOREM 5. *There exists a finite invariant measure μ with $\mu \sim m$ if and only if there exists a measurable set E with $0 < m_E(X) < \infty$.*

PROOF. By Corollary 2 the condition is necessary. Sufficiency is proved similarly in the proof of Theorem 4.

THEOREM 6. *If there exists a σ-finite invariant measure μ with $\mu \sim m$, then for any $\varepsilon > 0$ and any measurable set E with $m(E) > 0$, there exists a measurable set F such that $F \subset E$, $m(E-F) < \varepsilon$ and m_F is a σ-finite invariant measure with $m_F \sim m$.*

PROOF. Let $f = dm/d\mu$. Since $f(x) > 0$ a.e., for any $\varepsilon > 0$ there exists $a > 0$ such that

$$m(E \cap \{x: f(x) < a\}) < \varepsilon \quad \text{and} \quad m(E \cap \{x: f(x) \geq a\}) > 0.$$

Putting $F = E \cap \{x: f(x) \geq a\}$, we have

$$m(E-F) < \varepsilon \quad \text{and} \quad m(F) > 0.$$

Since by the definition of m_E and F

$$m_F(F) = \inf \left\{ \sum_{n=-\infty}^{\infty} m(T^n F_n): F = \bigcup_{n=-\infty}^{\infty} F_n,\ F_n \in \mathcal{F}, \right.$$
$$\left. T^n F_n \subset F \ (n = 0, \pm 1, \pm 2, \dots) \right\}$$

and by the Lemma

$$m(T^n F_n) = \int_{T^n F_n} f\, d\mu \geq a\mu(F_n) \quad \text{for } n = 0, \pm 1, \pm 2, \dots,$$

we have

$$m_F(F) \geq a\mu(F) > 0,$$

which implies $m_F \gg m$ by Corollary 3.

4. Finite invariant measures.

THEOREM 7. *Let μ and ν be finite invariant measures on $(X, \mathcal{F})$. If for any invariant measurable set E we have $\mu(E) = \nu(E)$, then $\mu = \nu$.*

PROOF. Let $\lambda(E) = (\mu(E) + \nu(E))/2$ for any measurable set E. Then we have $\lambda \gg \mu$ and $\lambda(E) = \mu(E)$ for any invariant measurable set E by assumption. By the Radon-Nikodým theorem there exists an integrable function f such that

$$\mu(E) = \int_E f\, d\lambda \tag{6}$$

for any measurable set E. Since we have

$$\mu(TE) = \int_E f(Tx)\, d\lambda(x)$$

and μ is invariant, we have by (6),

$$f(Tx) = f(x) \quad \text{a.e. } \lambda.$$

Therefore there exists an integrable function g such that $g(Tx) = g(x)$ for any x and $g(x) = f(x)$ a.e. λ. If $f = 1$ a.e. λ does not hold, there exists α ($\alpha < 1$ or $\alpha > 1$) such that an invariant measurable set

$$F = \{x: g(x) < \alpha\} \quad \text{or} \quad G = \{x: g(X) > \alpha\}$$

has λ-positive measure. In the former case $\mu(F) \leq \alpha\lambda(F)$, which contradicts assumption. The latter case is treated similarly.

The first half of Theorem 8 was proved by Hajian and Kakutani [**3**] using a Banach limit. We shall give a purely measure theoretic proof. This is also another construction of a finite invariant measure.

THEOREM 8. *There exists a finite invariant measure which is equivalent to m if and only if the following condition is satisfied.*

(A) *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(E) < \delta$, then $m(T^n E) < \varepsilon$ for $n = 0, \pm 1, \pm 2, \ldots$.*

If the condition (A) *is satisfied, then for any measurable set E the limit*

$$\mu(E) = \lim_{n\to\infty} \sum_{k=0}^{n-1} m(T^k E)$$

exists. μ is a finite invariant measure which is equivalent to m.

PROOF. It is easy to prove that (A) is satisfied if there exists a finite invariant measure which is equivalent to m. We prove that if (A) is satisfied, then the limit $\mu(E)$ exists and μ is a finite invariant measure which is equivalent to m. We put

$$\sigma_n(E) = \sum_{k=0}^{n-1} m(T^k E)$$

for any measurable set E and a positive integer n. We fix a measurable set E. Let $\mathcal{F}_0$ or $\mathcal{F}_{00}$ be a σ-algebra or algebra generated by $T^n E$ ($n = 0, \pm 1, \pm 2, \ldots$) respectively. $\mathcal{F}_{00}$ is countable. For any $F \in \mathcal{F}_0$ and for any $\varepsilon > 0$, there exists $G \in \mathcal{F}_{00}$ such that

$$m(F \cup G - F \cap G) < \varepsilon. \tag{7}$$

For each $F \in \mathcal{F}_{00}$, we consider a bounded sequence $(\sigma_n(F))$. Since $\mathcal{F}_{00}$ is countable, there exists a subsequence (σ_{n_i}) of the sequence (σ_n) of measures such that for any $F \in \mathcal{F}_{00}$, $(\sigma_{n_i}(F))$ is a Cauchy sequence. By (7) and assumption for any $F \in \mathcal{F}_0$, $(\sigma_{n_i}(F))$ is a Cauchy sequence. We define $\tilde{\mu}$ by

$$\tilde{\mu}(F) = \lim_{i\to\infty} \sigma_{n_i}(F)$$

for any $F \in \mathcal{F}_0$. $\tilde{\mu}$ is an additive set function on $(X, \mathcal{F}_0)$ but by assumption $\tilde{\mu}$ is a finite measure. It is easy to check that $\tilde{\mu}$ is invariant. The nonsingularity of T implies $m \gg \tilde{\mu}$. We prove that $\tilde{\mu} \gg m$. If $\tilde{\mu}(A) = 0$, then $\tilde{\mu}(A^*) = 0$. Since $\tilde{\mu}(A^*) = m(A^*)$ because A^* is invariant, we have $m(A) = 0$. Therefore the measure $\tilde{\mu}$ on $(X, \mathcal{F}_0)$ is finite, invariant and equivalent to m. If the limit $\lim_{n\to\infty} \sigma_n(E)$ does not exist, then $\liminf_{n\to\infty} \sigma_n(E) < \limsup_{n\to\infty} \sigma_n(E)$ and therefore there exist subsequences $(\sigma_{p_i}(E))$ and $(\sigma_{q_i}(E))$ of $(\sigma_n(E))$ converging to different limits:

$$\lim_{i\to\infty} \sigma_{p_i}(E) \neq \lim_{i\to\infty} \sigma_{q_i}(E). \tag{8}$$

By the procedure mentioned above we obtain two different finite invariant measures μ and ν on $(X, \mathcal{F}_0)$:

$$\mu(F) = \lim_{i\to\infty} \sigma_{r_i}(F) \quad \text{and} \quad \nu(F) = \lim_{i\to\infty} \sigma_{s_i}(F),$$

where (r_i) and (s_i) are subsequences of (p_i) and (q_i) respectively. For any invariant $\mathcal{F}_0$-measurable set G, $\mu(G) = \nu(G) = m(G)$. By Theorem 7, $\mu = \nu$, which contradicts (8).

From the above for any $E \in \mathcal{F}$ the limit

$$\mu(E) = \lim_{n\to\infty} \sigma_n(E)$$

exists. μ is an additive set function on $(X, \mathcal{F})$ but by assumption it is a finite measure on $(X, \mathcal{F})$. It is easy to see that μ is invariant. We can prove similarly as before that μ is equivalent to m.

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