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A NOTE ON THE g-FUNCTION

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ABSTRACT. We prove a pointwise inequality relating real-variable Littlewood-Paley g-functions and real-variable Lusin area functions.

Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be real and radial, $\operatorname{supp}\psi \subset \{|x| \leq 1\}$, $\int \psi = 0$. For $y > 0$ define $\psi_y(x) = y^{-d}\psi(x/y)$. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and $\alpha > 0$ we define

$$S^2_{\psi,\alpha}(f)(x) = \int_{|x-t|<\alpha y} |f * \psi_y(t)|^2 y^{-d-1}\,dt\,dy.$$

The function $S_{\psi,\alpha}(f)$ is a real-variable version of the familiar Lusin area function (as discussed, e.g., in [**1**]).

Let M denote the Hardy-Littlewood maximal function. In [**2**] it was proved that for all ψ as above, $\alpha > 0$, $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$, and $V \geq 0$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$, one has

$$\int S^2_{\psi,\alpha}(f)V\,dx \leq C \int |f|^2 MV\,dx, \tag{1}$$

where C is a constant that only depends on ψ, α, and d.

Let us define

$$g^2_\psi(f)(x) = \int_0^\infty |f * \psi_y(x)|^2 y^{-1}\,dy.$$

This is a real-variable version of the Littlewood-Paley g-function.

The function $g_\psi(f)$ is, intuitively, "smaller" (less singular) than $S_{\psi,\alpha}(f)$. Therefore, one should have

$$\int g^2_\psi(f)V\,dx \leq C(\psi, d) \int |f|^2 MV\,dx \tag{2}$$

for all f and V as above. Unfortunately, the proof of (1) does not carry over to yield a proof of (2). In this note, we show how to get (2) from (1). In particular, we prove

THEOREM. *Let ψ be as above. There is a real, radial $\rho \in C_0^\infty(\mathbf{R}^d)$*, $\operatorname{supp}\rho \subset \{|x| \leq 1\}$, $\int \rho = 0$, *such that*

$$g^2_\psi(f)(x) \leq C(d) S^2_{\rho,2}(f)(x) \tag{3}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and $x \in \mathbf{R}^d$.

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PROOF. Let $N > d/2 + 100$ be an integer and set $\rho = (1 - \Delta)^N \psi$, where Δ is the Laplacian. Then

$$\psi = \rho * g, \quad \text{where } \hat{g}(\xi) = (1 + |\xi|^2)^{-N}.$$

Clearly ρ is in $C_0^\infty(\mathbf{R}^d)$, is real and radial, and has support contained in $\{|x| \leq 1\}$. We only need to show (3). Now,

$$\psi(x) = \int \rho(x-t) g(t)\, dt.$$

By the support restriction on ρ and ψ, this integral is unchanged if we replace g by $h = g\chi_{\{|x| \leq 2\}}$. Therefore $\psi(x) = \rho * h$.

Let $f \in L^1_{\text{loc}}(\mathbf{R}^d)$: $f * \psi = f * \rho * h$. Since N is large, we have $|h| \leq C(d)$. Therefore

$$|f * \psi(x)| \leq C(d) \int_{|x-t|<2} |f * \rho(t)|\, dt \leq C(d) \left(\int_{|x-t|<2} |f * \rho(t)|^2\, dt \right)^{1/2}.$$

Thus,

$$|f * \psi(x)|^2 \leq C(d) \int_{|x-t|<2} |f * \rho(t)|^2\, dt.$$

This implies, by dilation invariance,

$$|f * \psi_y(x)|^2 \leq C(d) y^{-d} \int_{|x-t|<2y} |f * \rho_y(t)|^2\, dt,$$

i.e.,

$$\int_0^\infty |f * \psi_y(x)|^2 y^{-1}\, dy \leq C(d) \int_{|x-t|<2y} |f * \rho_y(t)|^2 y^{-d-1}\, dt\, dy$$

which is (3). Q.E.D.

REFERENCES

1. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.
2. S. Y. A. Chang, J. M. Wilson and T. H. Wolff, *Some weighted norm inequalities concerning the Schroedinger operators*, Comment. Math. Helv. **60** (1985), 217–246.

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