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COMPLETENESS THEOREM FOR SINGULAR BIPROBABILITY MODELS

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ABSTRACT. The aim of the paper is to prove the completeness theorem for singular biprobability models. This also solves Keisler's Problem 5.4 in the singular case (see [3]). The case of absolute continuity is considered in [6].

Let $\mathcal{A}$ be a countable admissible set and $\omega \in \mathcal{A}$. The logic $L_{P_1P_2\mathcal{A}}$ is similar to the standard probability logic $L_{\mathcal{A}P}$. The only difference is that two types of probability quantifiers ($P_1\vec{x} \geq r$ and $P_2\vec{x} \geq r$) are allowed.

The probability logic $L_{\mathcal{A}P}$ was introduced in [**2**] by H. J. Keisler. It is the logic which has formulas like those of $L_{\mathcal{A}} \subseteq L_{\omega_1\omega}$, except that quantifiers $P\vec{x} \geq r$ ($\vec{x}$ is a finite sequence of variables, $r \in [0,1]$ a real number) are used instead of the usual $\forall x$ and $\exists x$. A model is a probability space with measurable relations with respect to some extension of product measure. A formula $(P\vec{x} \geq r)\varphi(\vec{x})$ is true in the model if $\{\vec{x} : \varphi(\vec{x})\}$ has probability greater than r.

A singular biprobability model is a structure $(\mathfrak{A}, \mu_1, \mu_2)$ where

$$\mathfrak{A} = (A, R_i, c_j)_{i\in I, j\in J}$$

is a classical structure without operations and μ_1, μ_2 are types of probability measures such that μ_1 is singular with respect to μ_2, i.e. $\mu_1 \perp \mu_2$.

The quantifiers are interpreted in the natural way i.e.

$$(\mathfrak{A}, \mu_1, \mu_2) \models (P_1\vec{x} \geq r)\varphi(\vec{x}) \quad \text{iff } \mu_1^{(n)}\{\vec{x} \in A^n : (\mathfrak{A}, \mu_1, \mu_2) \models \varphi(\vec{x})\} \geq r$$

for $i = 1, 2$. (The measure $\mu_i^{(n)}$ is the restriction of the completion of μ_i^n to the σ-algebra generated by the measurable rectangles and the diagonal sets $\{\vec{x} \in A^n : x_i = x_j\}$.)

Axioms and rules of inference are those of $L_{\mathcal{A}P}$, as listed in [**1**] in conjunction with the axiom B_4 from [**3**] with the remark that both P_1 and P_2 can play the role of P, together with the following axioms:

Axioms of continuity.

$$(1) \qquad \bigwedge_{n\geq 1} \bigvee_{m\geq 1} \left(P_i\vec{y} < \frac{1}{n}\right)\left(P_j\vec{x} \in \left[r - \frac{1}{m}, r\right)\right)\varphi(\vec{x}, \vec{y}), \qquad i, j = 1, 2,$$

$$(2) \qquad \bigwedge_{n\geq 1} \bigvee_{m\geq 1} \left(P_i\vec{y} < \frac{1}{n}\right)\left(P_j\vec{x} \in \left(r, r + \frac{1}{m}\right]\right)\varphi(\vec{x}, \vec{y}).$$

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Axioms of singularity.

$$(P_i x = 0)((P_1 y > 0)(x = y) \wedge (P_2 y > 0)(x = y)), \qquad i = 1, 2.$$

In order to prove the main result, let us introduce two sorts of auxiliary models.

DEFINITION. (i) A *weak structure* for $L_{P_1P_2\mathcal{A}}$ is a structure $(\mathfrak{A}, \mu_{n,i})_{n\geq 1, i=1,2}$ (shortly $(\mathfrak{A}, \mu_{n,i})$) such that each $\mu_{n,i}$ is a finitely additive probability measure on A^n with each singleton measurable and the set $\{\vec{b} \in A^n\colon (\mathfrak{A}, \mu_{n,i}) \models \varphi[\vec{a}, \vec{b}]\}$ is $\mu_{n,i}$-measurable for each $\varphi(\vec{x}, \vec{y}) \in L_{P_1P_2\mathcal{A}}$ and $\vec{a} \in A^m$.

(ii) A *middle structure* for $L_{P_1P_2\mathcal{A}}$ is a weak structure $(\mathfrak{A}, \mu_{n,i})$ such that the following is true:

There is a set $B \subseteq A$ such that $\mu_1(B) = 0$ and $\mu_2(B) = 1$.

In both cases the concept of satisfaction is defined in a natural way.

LEMMA 1. *A theory T of $L_{P_1P_2\mathcal{A}}$ is consistent if and only if it has a weak model in which each theorem of $L_{P_1P_2\mathcal{A}}$ is true.*

The proof involves the Henkin construction similar to that in [**1**].

In order to prove the next lemma, we use the following Los-Marczewski theorem.

THEOREM 1 (SEE [**4**], [**5**]). *Let C be a field of subsets of a set Ω. Let μ be a finitely additive probability measure on C. Let $A \subseteq \Omega$ be such that $A \notin C$. Let $\mathcal{F}(C, A)$ be the smallest field on Ω containing C and A. Then there exists a finitely additive probability measure $\bar{\mu}$ on $\mathcal{F}(C, A)$ which is an extension of μ.*

Moreover, if d is any real number between

$$\mu_i(A) = \sup\left\{\frac{\sum_{j=1}^n \mu(A_j) - \sum_{j=1}^m \mu(B_j)}{k} : A_1, \ldots, A_n, B_1, \ldots, B_m,\; kI_A + \sum_{j=1}^m I_{B_j} \geq \sum_{j=1}^n I_{A_i}\right\}$$

and

$$\mu_e(A) = \inf\left\{\frac{\sum_{j=1}^n \mu(A_j) - \sum_{j=1}^m \mu(B_j)}{k} : A_1, \ldots, A_n,\; B_1, \ldots, B_m,\; \sum_{j=1}^n I_{A_j} \geq \sum_{j=1}^m I_{B_j} + kI_A\right\},$$

then there is a finitely additive probability measure $\bar{\mu}$ on $\mathcal{F}(C, A)$ such that $\bar{\mu}(A) = d$ and $\bar{\mu}$ is an extension of μ from C to $\mathcal{F}(C, A)$. (I_A is an indicator function of the set A.)

LEMMA 2. *A theory T of $L_{P_1P_2\mathcal{A}}$ is consistent if and only if it has a middle model in which each theorem of $L_{P_1P_2\mathcal{A}}$ is true.*

PROOF. The nontrivial part is to prove that if T is consistent then T has a middle model.

Let $(\mathfrak{A}, \mu_{n,i})$ be a weak model of T in which each theorem of $L_{P_1P_2\mathcal{A}}$ is true.

Let $Z_0, Z_1, \ldots$ enumerate $\operatorname{dom}\mu_{1,1} = \operatorname{dom}\mu_{1,2}$.

Let us choose sets U and S in the following way:

$$U_0 = \{x \in A\colon \mu_{1,1}(\{x\}) > 0\}, \quad S_0 = \{x \in A\colon \mu_{1,2}(\{x\}) > 0\},$$
$$U_{n+1} = U_n \cup \{x\}, \quad \text{where } x \in Z_n \backslash (U_n \cup S_n) \quad \text{(if there is any)},$$
$$S_{n+1} = S_n \cup \{x\}, \quad \text{where } x \in Z_n \backslash (U_{n+1} \cup S_n) \quad \text{(if there is any)}.$$

Let $U = \bigcup_{n\geq 0} U_n$ and $S = \bigcup_{n\geq 0} S_n$. It is easy to see that $U \cap S = \varnothing$ and that each set $Z_n \in \operatorname{dom}\mu_{1,1} = \operatorname{dom}\mu_{1,2}$ with a positive measure $\mu_{1,1}$ ($\mu_{1,2}$) has a nonempty intersection with U (U^c).

Let us show that $(\mu_{1,1})_e(U) = 1$. (Let us abbreviate $(\mu_{1,1})_e$ as μ in the rest of the proof.)

Let $A_1, \dots, A_n$, $B_1, \dots B_m$, be members from $\operatorname{dom}\mu$ such that

$$\sum_{i=1}^{n} I_{A_i} \geq \sum_{i=1}^{m} I_{B_i} + kI_A.$$

Let $C_s = \{x \in A\colon \sum_{i=1}^{n} I_{A_i}(x) = s\}$, $s = 0, \dots, n$, $D_s = \{x \in A\colon \sum_{i=1}^{m} I_{B_i}(x) = s\}$, $s = 0, \dots, m$.

It is easy to see that $\mu(D_s) = 0$ for $s > n - k$ and $\mu(C_s) = 0$ for $s < k$. Consequently $\mu(D_{n-k}) + \cdots + \mu(D_{n-k-i}) \leq \mu(C_n) + \cdots + \mu(C_{n-i})$ for $i = 0, \dots, n, k-1$.

It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \mu(A_i) - \sum_{i=1}^{m} \mu(B_i) &= \sum_{s=1}^{n} s\mu(C_s) - \sum_{s=1}^{m} s\mu(D_s) \\
&= \sum_{s=k}^{n} s\mu(C_s) - \sum_{s=1}^{n-k} s\mu(D_s) = \sum_{s=k}^{n} s\mu(C_s) - \sum_{s=k}^{n} (s-k)\mu(D_{s-k}) \\
&\geq \sum_{s=k}^{n} s\mu(C_s) - \sum_{s=k}^{n} (s-k)\mu(C_s) = \sum_{s=k}^{n} k\mu(C_s) = k\sum_{s=0}^{n} \mu(C_s) = k.
\end{aligned}$$

So, $\mu_e(U) = 1$ or in precise notation $(\mu_{1,1})_e(U) = 1$.

Similarly it can be shown that $(\mu_{1,2})_e(U^c) = 1$.

Hence by Theorem 1, measures $\mu_{1,1}$ and $\mu_{1,2}$ can be extended so that $\mu_{1,1} \subseteq \bar{\mu}_{1,1}$, $\mu_{1,2} \subseteq \bar{\mu}_{1,2}$ and $\bar{\mu}_{1,1} \perp \bar{\mu}_{1,2}$.

LEMMA 3. *For each weak model $(\mathfrak{A}, \mu_{n,i})$ in which each theorem of $L_{P_1P_2\mathcal{A}}$ is true there is a singluar biprobability model $(\mathfrak{A}, \mu_1, \mu_2)$ such that $(\mathfrak{A}, \mu_{n,i}) \equiv (\mathfrak{A}, \mu_1, \mu_2)$.*

The proof of the lemma makes use of Loeb-Hoover-Keisler construction (see [**3**]) and Lemma 2.

THEOREM 2 (COMPLETENESS THEOREM FOR $L_{P_1P_2\mathcal{A}}$ LOGIC). *A theory T of $L_{P_1P_2\mathcal{A}}$ is consistent if and only if T has a singular biprobability model.*

The proof follows easily from Lemmas 1, 2 and 3.

BIBLIOGRAPHY

1. D. Hoover, *Probability logic*, Ann. Math. Logic **14** (1978), 287–313.
2. H. J. Keisler, *Hyperfinite model theory*, Logic Colloquium '76, edited by J. M. E. Hyland and R. O. Gandy, North-Holland, 1977, pp. 5–110.
3. ———, *Probability quantifiers*, Chapter 14 in Model Theoretic Logics, edited by J. Barwise and S. Feferman, Springer-Verlag, 1985, pp. 509–556.
4. J. Los and E. Marczewski, *Extension of measures*, Fund. Math. **36** (1949), 267–276.
5. K. P. S. Rao Bhaskara and M. Rao Bhaskara, *Theory of charges*, Academic Press, 1983.
6. M. D. Rašković, *Completeness theorem for biprobability models*, J. Symbolic Logic **51** (1986), 586–590.

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