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VECTOR-VALUED STOCHASTIC PROCESSES. II. A RADON-NIKODÝM THEOREM FOR VECTOR-VALUED PROCESSES WITH FINITE VARIATION

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ABSTRACT. Given a real-valued process A with finite variation $|A|$ and a vector-valued process B with finite variation $|B|$ such that for each ω, the Stieltjes measure $dB.(\omega)$ is absolutely continuous with respect to $dA.(\omega)$, there exists a vector-valued process H which, under certain separability conditions, satisfies $B_t = \int_{[0,t]} H_s\, dA_s$ and $|B|_t = \int_{[0,t]} \|H_s\| d|A|_s$ for every $t \geq 0$. If, moreover, A and B are optional or predictable, then so is H.

1. Introduction. This note was inspired by [**1**, VI, 68], where a Radon-Nikodým theorem is proved for real-valued processes with finite variation. We extend this theorem for stochastic functions with finite variation which are not necessarily measurable, with values in spaces $L(E,F)$. This result is used in [**4**] to prove a characterization of optional processes with finite variation.

We shall use the definitions and notations in [**1**]. Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration, that is, an increasing family of sub-σ-algebras of $\mathcal{F}$. We assume that the filtration satisfies the usual conditions, that is, $\mathcal{F}_0$ contains all P-negligible sets of $\mathcal{F}$, and $\mathcal{F}_t = \bigcap\{\mathcal{F}_s; s > t\}$ for every $t \geq 0$. Let E, F be Banach spaces and Z a subspace of F', norming for F, that is, such that $\|y\| = \sup\{|\langle y, z\rangle|; z \in Z, \|z\| \leq 1\}$ for every $y \in F$.

We shall consider stochastic functions $X = (X_t)_{t\geq 0}$ defined on $R_+ \times \Omega$ with values in $L(E,F)$ or E. We do not assume that each X_t is $\mathcal{F}$-measurable; if each X_t is $\mathcal{F}$-measurable, X is called a stochastic process. If, moreover, X_t is $\mathcal{F}_t$-measurable for each t, X is called an adapted process. The function $\omega \to \|X_t(\omega)\|$ will be denoted by $\|X_t\|$ and the stochastic function $(\|X_t\|)_{t\geq 0}$ will be denoted by $\|X\|$. We shall denote by $|X(\omega)|_{[0,t]}$ or $|X|_{[0,t]}(\omega)$ the variation of the function $s \to X_s(\omega)$ on $[0,t]$. We set $|X|_t(\omega) = \|X_0(\omega)\| + |X|_{[0,t]}(\omega)$. The stochastic function $|X| = (|X|_t)_{t\geq 0}$ is called the variation of X. If X is scalar-valued, we distinguish between $|X_t|$ and the variation $|X|_t$ and it will be clear from the context whether $|X|$ denotes $(|X_t|)_{t\geq 0}$ or $(|X|_t)_{t\geq 0}$. A process X is said to be a raw increasing process if for each $\omega \in \Omega$, the path $t \to X_t(\omega)$ is increasing and right continuous and $X_0(\omega) = 0$. If, moreover, X is adapted, it is called, simply, an increasing process. We note that a process with finite variation is right continuous if and only if its variation is right continuous (see [**5**, Appendix]). A process X is called measurable, if it is $\mathcal{B}(\mathbf{R}_+) \times \mathcal{F}$-measurable. X is called optional, if it is measurable with respect to the σ-algebra generated by

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the adapted cadlag processes (that is processes that are right continuous and have left limits as function of t, for each ω). X is called predictable if it is measurable with respect to the σ-algebra generated by the adapted left continuous processes. A set $A \subset R_+ \times \Omega$ is called evanescent, if it is contained in a set $\mathbf{R}_+ \times F$ with $F \in \mathcal{F}$ and $P(F) = 0$. The expressions "almost surely" (a.s.) and "negligible" are understood to be with respect to P.

2. The main result. We collect all the results in the following theorem.

THEOREM. *Let A be a real-valued right continuous process with $A_{0-} = 0$ and B: $R_+ \times \Omega \to L(E, F)$ a right continuous stochastic function with $B_{0-} = 0$ satisfying the following conditions*:

(i) *The variations $|A|$ and $|B|$ of A and B are raw, finite, increasing processes*;

(ii) *For every $x \in E$ and $z \in Z$, $\langle Bx, z\rangle$ is a raw process with finite variation*;

(iii) *For every $\omega \in \Omega$, the measure $dB.(\omega)$ is absolutely continuous with respect to $dA.(\omega)$.*

Then there exists a stochastic function H: $R_+ \times \Omega \to L(E, Z')$ satisfying the following conditions:

(1) *For every $x \in E$ and $z \in Z$ there is a negligible set in $\mathcal{F}$ outside of which the function $\langle H.(\omega)x, z\rangle$ is $dA.(\omega)$-integrable on $[0, t]$ for every $t \geq 0$ and*

$$\langle B_t(\omega)x, z\rangle = \int_{[0,t]} \langle H_s(\omega)x, z\rangle \, dA_s(\omega).$$

Moreover, for any measurable process $X \in L^1_E(\mu_{|B|})$, the function $X.(\omega)$ is $dB.(\omega)$-integrable a.s., for every $z \in Z$ the function $\langle H.(\omega)X.(\omega), z\rangle$ is $dA.(\omega)$ integrable a.s. and

$$\left\langle \int_{[0,\infty)} X_s dB_s, z \right\rangle = \int_{[0,\infty)} \langle H_s X_s, z\rangle \, dA_s \quad a.s.$$

(2) *There is a negligible set in $\mathcal{F}$ outside of which the function $||H.(\omega)||$ is $d|A|.(\omega)$-integrable on $[0, t]$ for every $t \geq 0$ and*

$$|B|_t \geq \int_{[0,t]} ||H_s|| \, d|A|_s \quad a.s.$$

(3) *We can choose H with values in $L(E, F)$ in each of the following cases*:

(a) *F is the dual of a Banach space G and $Z = G$.*

(b) *There is a sequence (Ω_m) of disjoint not negligible sets of $\mathcal{F}$ with union Ω, such that for every $n, m \in N$ and $x \in E$, the convex equilibrated cover of the set $\{B_t(\omega)x; (t, \omega) \in [0, n] \times \Omega_m\}$ is $\sigma(F, Z)$-relatively compact.*

(c) *E is separable and F has the Radon-Nikodým property. In this case Hx is locally μ_A-integrable (that is, μ_A-integrable on $[0, t] \times \Omega$ for $t \geq 0$) for every $x \in E$ and*

$$B_t x = \int_{[0,t]} H_s x \, dA_s.$$

(d) *B takes on values in a subspace $G \subset L(E, F)$ having the Radon-Nikodým property. In this case H is locally μ_A-integrable and*

$$B_t = \int_{[0,t]} H_s \, dA_s.$$

(4) *We have the equality*

$$|B|_t = \int_{[0,t]} ||H_s||\, d|A|_s$$

(*except on an evanescent set*) *in each of the following cases*:

(α) *There is a lifting* ρ *of* P *such that* $\rho[B_t] = B_t$ *for* $t \geq 0$, *i.e., there exists a sequence* (C_n) *in* $\mathcal{F}$ *with* $\Omega - \bigcup C_n$ *negligible, such that for every* $x \in E$ *and* $z \in Z$ *we have* $1_{C_n}\langle B_t x, z\rangle \in L^\infty(P)$ *and*

$$\rho(1_{C_n}\langle B_t x, z\rangle) = 1_{\rho(C_n)}\langle B_t x, z\rangle;$$

(β) E *is separable and there is a separable subspace* $S \subset Z$, *norming for* F;

(γ) E *is separable and for every* $x \in E$ *and* $t \geq 0$, $B_t x$ *is separably valued*;

(δ) *For every* $t \geq 0$, B_t *is separably valued.*

(5) *If* A, $|B|$ *and* $\langle Bx, z\rangle$ *are measurable* (*respectively optional, predictable*) *for* $x \in E$ *and* $z \in Z$ *and if condition* (β) *is satisfied, then* H *can be chosen such that* $\langle Hx, z\rangle$ *is measurable* (*respectively optional, predictable*) *for* $x \in E$ *and* $z \in S$.

If, in addition, condition (γ) *is satisfied, then* Hx: $R_+ \times \Omega \to Z'$ *is measurable* (*respectively optional, predictable*) *for every* $x \in E$. *If, further, condition* (δ) *is satisfied, then* H: $R_+ \times \Omega \to L(E, Z')$ *is measurable* (*respectively optional, predictable*).

PROOF. Let $\mathcal{M}$ denote the σ-field $\mathcal{B}(R_+) \times \mathcal{F}$ and $\mathcal{M}_b$ denote the δ-ring of the sets of $\mathcal{M}$ contained in some set of the form $[0, t] \times \Omega$.

We shall assume first that $|A|_t$ and $|B|_t$ are integrable for all $t \geq 0$.

Let μ_A: $\mathcal{M}_b \to R$ be the measure vanishing on evanescent sets, corresponding to A; that is, satisfying for every $M \in \mathcal{M}_b$ (see [**3**])

$$\mu_A(M) = E\left(\int 1_M\, dA_s\right)$$

and

$$|\mu_A|(M) = \mu_{|A|}(M) = E\left(\int 1_M d|A|_s\right).$$

We define similarly the measure $\mu_{|B|}$ by

$$\mu_{|B|}(M) = E\left(\int 1_M d|B|_s\right) \quad \text{for } M \in \mathcal{M}_b.$$

For $x \in E$ and $z \in Z$, the variation $|\langle Bx, z\rangle|$ of $\langle Bx, z\rangle$ satisfies $|\langle Bx, z\rangle|_t \leq |B|_t ||x||\, ||z||$; therefore, $|\langle Bx, z\rangle|_t$ is integrable for $t \geq 0$; we can then define the measure $m_{x,z}$: $\mathcal{M}_b \to R$ vanishing on evanescent sets and satisfying for every $M \in \mathcal{M}_b$

$$m_{x,z}(M) = E\left(\int 1_M d\langle B_s x, z\rangle\right).$$

For $M \in \mathcal{M}_b$ the mapping $(x, z) \to m_{x,z}(M)$ is bilinear and satisfies $||m_{x,z}(M)|| \leq \mu_{|B|}(M)||x||\, ||z||$. Then there exists a continuous linear mapping $m(M) \in L(E, Z')$ satisfying

$$\langle m(M)x, z\rangle = m_{x,z}(M) = E\left(\int 1_M d\langle B_s x, z\rangle\right)$$

and $||m(M)|| \leq \mu_{|B|}(M)$. It follows that m: $\mathcal{M}_b \to L(E, Z')$ is a σ-additive measure with σ-finite variation $|m|$ satisfying $|m| \leq \mu_{|B|}$. From [**3**, Theorem 5] we deduce that if $X \in L^1_E(\mu_{|B|})$ then

$$\langle m(X), z\rangle = E\left(\left\langle\int X_s\, dB_s, z\right\rangle\right) \quad \text{for } z \in Z.$$

Hypothesis (iii) implies that $m << \mu_A$. By the Radon-Nikodým theorem there is a locally μ_A-integrable function ϕ such that $|m| = \phi\mu_A$, that is,

$$|m|(M) = \int_M \phi\, d\mu_A \quad \text{for } M \in \mathcal{M}_b.$$

We have also $|m| = |\phi\mu_A| = |\phi|\,|\mu_A| = |\phi|\mu_{|A|}$, that is,

$$|m|(M) = \int_M |\phi|\, d\mu_{|A|} \quad \text{for } M \in \mathcal{M}_b.$$

We can apply the extended Radon-Nikodým theorem [**2**, Theorem 5, p. 269] to m and $|m|$ and find a function H': $R_+ \times \Omega \to L(E, Z')$ satisfying the following conditions:

(1′) For every $x \in E$, $z \in Z$, and $M \in \mathcal{M}_b$ the function $\langle H'x, z\rangle$ is $|m|$-integrable over M and

$$\langle m(M)x, z\rangle = \int_M \langle H'x, z\rangle\, d|m|.$$

(2′) $||H'|| = 1$, $|m|$-a.e.

(3′) For every $X \in L^1_E(|m|)$ and $z \in Z$, the function $\langle H'X, z\rangle$ is $|m|$-integrable, X is m-integrable and

$$\langle m(X), z\rangle = \int \langle H'x, z\rangle\, d|m|.$$

(4′) If ρ_B is a lifting of $|m|$ we can choose H' uniquely everywhere such that $\rho_B(H') = H'$, that is, $\rho_B(\langle H'x, z\rangle) = \langle H'x, z\rangle$ for $x \in E$ and $z \in Z$. Moreover, we can take ρ_B such that $\rho_B([0,n] \times \Omega_m) = [0,n] \times \Omega_m$ for every n and m, where Ω_m are as in assertion (3b).

We can realize this by applying the extended Radon-Nikodým theorem to the measures m and $|m|$ restricted to the σ-algebra $\mathcal{M}_{0,m} = \mathcal{M} \cap ([0,1] \times \Omega_m)$ of $[0,1] \times \Omega_m$ and to the σ-algebras $\mathcal{M}_{n,m} = \mathcal{M} \cap ((n, n+1] \times \Omega_m)$ of $(n, n+1] \times \Omega_m$ for $n, m \in N$ and obtain functions $K^{n,m}$ ($n \geq 0$, $m \geq 1$) satisfying, for $M \in \mathcal{M}_b$,

$$\langle m(M)x, z\rangle = \int_M \langle K^{n,m}x, z\rangle\, d|m|,$$

and $\rho_{n,m}(K^{n,m}) = K^{n,m}$ for a lifting $\rho_{n,m}$ of $|m|$ restricted to $[0,1] \times \Omega_m$, respectively to $(n, n+1] \times \Omega_m$, and $||K^{mn}|| \equiv 1$.

We set then

$$H' = \sum_{m\geq 1} K^{0,m} 1_{[0,1]} + \sum_{n,m\geq 1} K^{n,m} 1_{(n,n+1]}$$

and

$$\rho_B(f) = \sum_{m\geq 1} \rho_{0,m}(f 1_{[0,1]\times\Omega_m}) + \sum_{n,m\geq 1} \rho_{n,m}(f 1_{(n,n+1]\times\Omega_m})$$

for $f \in L^\infty(|m|)$. If we denote $H = H'\phi$ we deduce that

(1″) For every $x \in E$, $z \in Z$, and $M \in \mathcal{M}_b$ the function $\langle Hx, z\rangle$ is $\mu_{|A|}$-integrable over M and

$$\langle m(M)x, z\rangle = \int_M \langle Hx, z\rangle \, d\mu_A.$$

(2″) For every $M \in \mathcal{M}_b$, $||H||$ is $\mu_{|A|}$-integrable over M and

$$|m|(M) = \int_M ||H|| \, d\mu_{|A|}.$$

(3″) For any $X \in L^1_E(|m|)$, X is m-integrable; for every $z \in Z$, $\langle HX, z\rangle$ is μ_A-integrable and

$$\langle m(X), z\rangle = \int \langle HX, z\rangle \, d\mu_A = E\left(\int \langle H_s X_s, z\rangle \, dA_s\right).$$

We could have obtained H by applying directly the extended Radon-Nikodým theorem to m and μ_A; but we shall need property (4′) to prove assertion (3b).

From the two representations of $\langle m(X), z\rangle$ we deduce

$$E\left(\left\langle \int X_s \, dB_s, z\right\rangle\right) = E\left(\int \langle H_s X_s, z\rangle \, dA_s\right) \quad \text{for } X \in L^1_E(|m|).$$

Replacing X by $1_F X$ with $F \in \mathcal{F}$, we deduce then

$$\left\langle \int X_s \, dB_s, z\right\rangle = \int \langle H_s X_s, z\rangle \, dA_s \quad \text{a.s. for } X \in L^1_E(|m|).$$

In particular, for $X = 1_{[0,t]}x$ or $X = 1_{(r,t]}x$ with $x \in E$, we deduce

$$\langle B_t x, z\rangle = \int_{[0,t]} \langle H_s x, z\rangle \, dA_s \quad \text{a.s.}$$

and

$$\langle (B_t - B_r)x, z\rangle = \int_{(r,t]} \langle H_s x, z\rangle \, dA_s \quad \text{a.s.}$$

From the representations of $|m|(M)$ and $\mu_{|B|}(M)$ and from $|m| \le \mu_{|B|}$ we obtain

$$E\left(\int 1_M \, d|B|_s\right) \ge E\left(\int 1_M ||H_s|| \, d|A|_s\right) \quad \text{for } M \in \mathcal{M}_b.$$

Taking $M = [0,t] \times F$ or $M = (r,t] \times F$ with $F \in \mathcal{F}$ we get

$$|B|_t \ge \int_{[0,t]} ||H_s|| \, d|A|_s \quad \text{a.s.}$$

and

$$|B|_t - |B|_r \ge \int_{(r,t]} ||H_s|| \, d|A|_s \quad \text{a.s.}$$

The representation of $B_t - B_r$ will be needed in the last part of the proof, when $|A|_t$ and $|B|_t$ will not be assumed integrable. By right continuity, the above relations are valid outside an evanescent set, and this proves assertions (1) and (2). In case both A and B are scalar-valued and measurable (respectively optional, predictable), then by [**1**, VI, 68] H can be chosen measurable (respectively optional, predictable).

We prove now assertion (3). Case (a) is trivial. Consider case (b).

Let $x \in E$ and denote by $C_{n,m}$ the closed convex equilibrated cover of the set $\{B_t(\omega)x;\ (t,\omega) \in [0,n] \times \Omega_m\}$. Since $C_{n,m}$ is $\sigma(F,Z)$-compact, there is a family $(z_i)_{i\in I}$ in Z such that $C_{n,m} = \bigcap_{i\in I}\{y \in Z^*;\ |\langle y, z_i\rangle| \leq 1\}$, where Z^* is the algebraic dual of Z. Then $|\langle B_t(\omega)x, z_i\rangle| \leq 1$ for all $(t,\omega) \in [0,n] \times \Omega_m$ and $i \in I$. For $0 \leq t \leq n$ and $F \in \mathcal{F} \cap \Omega_m$ we have $|\langle m([0,t] \times F)x, z_i\rangle| \leq E(1_F|\langle B_t x, z\rangle|) \leq 1$; therefore, $m([0,t] \times F)x \in C_{n,m} \subset F$ (it follows, in particular, that m has values in $L(E,F)$). Then

$$\int_{[0,t]\times F} \langle H'_s x, z_i\rangle \, d|m| = |\langle m([0,t] \times F)x, z_i\rangle| \leq 1$$

for all $i \in I$. It follows that

$$|\langle H'_t(\omega)x, z_i\rangle| \leq 1, \quad |m|\text{-a.e. on } [0,n] \times \Omega_m.$$

Since $\rho_B(\langle H'x, z_i\rangle) = \langle H'x, z_i\rangle$ and $\rho_B([0,n] \times \Omega_m) = [0,n] \times \Omega_m$, we deduce $|\langle H'_t(\omega)x, z_i\rangle| \leq 1$ everywhere on $[0,n] \times \Omega_m$ for all $i \in I$. It follows that for $(t,\omega) \in [0,n] \times \Omega_m$ we have $H'_t(\omega)x \in C_{n,m} \subset F$; therefore, $H_t(\omega) = H'_t(\omega)\phi_t(\omega) \in F$ and this proves (b).

Assume now that E is separable and F has the RNP and prove assertion (3c). Let $x \in E$. The measure m_x: $\mathcal{M}_b \to F$ defined by $m_x(M) = m(M)x$ for $M \in \mathcal{M}_b$ is σ-additive and has σ-finite variation satisfying $|m_x| \leq |m|\,||x||$; therefore $|m_x| << \mu_A$. Since F has the RNP, there is an F-valued, measurable locally μ_A-integrable function H_x such that $m_x(M) = \int_M H_x \, d\mu_A$ for $M \in \mathcal{M}_b$. We choose H_x separably valued, therefore we can assume F separable, and we can choose Z separable in F' and norming for F. Consider the stochastic function H constructed for this choice of Z. We have then for z in a countable dense subset $Z_0 \subset Z$ and $M \in \mathcal{M}_b$

$$\int_M \langle H_x, z\rangle \, d\mu_A = \langle m_x(M), z\rangle = \langle m(M)x, z\rangle = \int_M \langle Hx, z\rangle \, d\mu_A;$$

therefore $\langle H_x, z\rangle = \langle Hx, z\rangle$, μ_A-a.e. Since Z_0 is countable we deduce that $Hx = H_x \in F$, μ_A-a.e. Moreover, since E is separable, the exceptional set is independent of x, hence modifying H on a μ_A-negligible set, we can get H with values in $L(E,F)$ everywhere such that Hx is locally μ_A-integrable for every $x \in E$, and this proves (3c). Part (d) of assertion (3) is evident.

Assertion (4) follows from the fact proved in [**3**] that in all cases α–δ we have $|m| = \mu_{|B|}$.

To prove assertion (5) let Σ be one of the σ-fields $\mathcal{M}$, $\mathcal{O}$, $\mathcal{P}$, according to whether A, $|B|$, and $\langle Bx, z\rangle$ are, respectively, measurable, optional or predictable. Consider first case (β): E is separable and there is a separable subspace $S \subset Z$ norming for F. For every $x \in E$ and $z \in S$ we apply the scalar version of the present theorem [**1**, V1, 68] and obtain a Σ-measurable scalar process $H_{x,z}$ such that

$$E\left(\int |H_{x,z}| \, d|A|_s\right) < \infty$$

and

$$\langle B_t x, z\rangle = \int_{[0,t]} H_{x,z} \, dA_s$$

except on an evanescent set from Σ. It follows that

$$\int_{[0,t]} \langle H_s x, z\rangle \, dA_s = \int_{[0,t]} H_{x,z} \, dA_s$$

except on an evanescent set $R_+ \times N_{x,z}$ with $N_{x,z} \subset \Omega$ negligible. For $\omega \notin N_{x,z}$ we have then $\langle H_t(\omega)x, z\rangle = H_{x,z}(t,\omega)$ except on a $d|A|.(\omega)$-negligible subset of R_+. Let $C = \{(t,\omega); \langle H_t(\omega)x,z\rangle \neq H_{x,z}(t,\omega)\}$. For $\omega \notin N_{x,z}$ the section $C(\omega) = \{t; \langle H_t(\omega)x, z\rangle \neq H_{x,z}(t,\omega)\}$ is $d|A|.(\omega)$-negligible, hence,

$$\mu_A(C) = E\left(\int 1_C \, d|A|_s\right) = 0,$$

that is, $\langle Hx, z\rangle = H_{x,z}$ except on a μ_A-negligible set of Σ depending on x and z. Let E_0 and S_0 be countable dense subsets of E and S respectively. We can alter H and $H_{x,z}$ on a μ_A-negligible set of Σ such that $\langle Hx, z\rangle = H_{x,z}$ everywhere for $x \in E_0$, $z \in S_0$. It follows that $\langle Hx, z\rangle$ is Σ-measurable for $x \in E_0$ and $z \in S_0$, and then, by taking limits, for all $x \in E$ and $z \in S$.

Consider now the case (γ). Since $B_t x$ is separably valued for $x \in E$ and $t \geq 0$, by right continuity we deduce that Bx is separably valued; since E is separable, we can assume that F is separable and we can take a separable subset $S \subset Z$ norming for F. Then, by the case (β), $\langle Hx, z\rangle$ is Σ-measurable for every $x \in E$ and $z \in S$, hence, by [**2**, Proposition 22, p. 105], Hx is Σ-measurable for every $x \in E$.

Finally, consider case (δ): B_t is separably valued for $t \geq 0$. By right continuity B is separably valued, therefore we can assume that E is separable. By case (β), Hx is Σ-measurable for every $x \in E$, therefore by [**2**, Proposition 18, p. 102], H is Σ-measurable.

We remark that in [**2**], Propositions 12 and 18 quoted above are proved for measurability with respect to a measure, but the statements and the proofs remain true for measurability with respect to a σ-algebra (see Appendix in [**4**]).

This proves the theorem in case $|A|_t$ and $|B|_t$ are integrable for all $t \geq 0$. Assume now that $|A|_t$ and $|B|_t$ are only finite, but not necessarily integrable for all $t \geq 0$. Using the idea of [**1**, VI, 68 bis] we replace P with equivalent probabilities such that $|A|_t$ and $|B|_t$ become integrable.

For every $n \in N$ consider the processes A^n and B^n obtained from A and B stopped at n. Let $Q_n = c_n(1 + |A|_n + |B|_n)P$ where c_n is a constant chosen such that $Q_n(1) = 1$. Then, for $t \leq n$, $|A|_t$ and $|B|_t$ are Q_n-integrable, that is, $|A^n|_t = |A|_t^n$ and $|B^n|_t = |B|_t^n$ are Q_n-integrable for all $t \geq 0$. We remark that P and Q_n are equivalent, therefore the evanescent sets are the same for P and Q_n.

By the above part of the proof there exists a stochastic function $H^n\colon R_+ \times \Omega \to L(E, Z')$ satisfying all the conclusions of the theorem. In particular, for $x \in E$, $z \in Z$, and $0 \leq t \leq n$ we have

$$\langle B_t x, z\rangle = \int_{[0,t]} \langle H_s^n x, z\rangle \, dA_s$$

and

$$|B|_t \geq \int_{[0,t]} ||H_s^n|| \, dA_s$$

with equality if one of the conditions α or β is satisfied. Similarly, for $0 \leq r < t \leq n$ we have

$$\langle (B_t - B_r)x, z\rangle = \int_{(r,t]} \langle H_s^n x, z\rangle \, dA_s$$

and

$$|B|_t - |B|_r \geq \int_{(r,t]} ||H_s^n|| \, dA_s,$$

with equality if α or β is satisfied. We take then

$$H = H^1 1_{[0,1]} + \sum_{n \geq 1} H^{n+1} 1_{(n,n+1]}.$$

Then H satisfies all the conclusions 1–5 of the statement and the theorem is completely proved.

REMARKS. (1) Under one of the conditions α–δ of assertion (4), the assumption in hypothesis (i) that the variation $|B|$ is measurable follows automatically from hypothesis (ii) (see Theorems 4 and 5 in [**5**]). Moreover, under condition α, if $\langle Bx, z\rangle$ is optional for every $x \in E$ and $z \in Z$, then $|B|$ is optional; under one of the conditions β–δ, if $\langle Bx, z\rangle$ is optional or predictable for every $x \in E$ and $z \in Z$, then $|B|$ has the same property.

(2) Assertions (1), (2), and (3) of the theorem remain valid if in hypothesis (i) the assumption that $|B|$ is measurable is replaced by the condition that there exists a raw, finite, increasing process D satisfying $|B|_0 \leq D_0$ and $|B|_t - |B|_s \leq D_t - D_s$ for $s \leq t$. In this case we replace $|B|_t$ by D_t in the inequality of asssertion 2 and in the proof.

3. Particular cases. The most important particular case is when B takes on values in a given Banach space and $|B|$ is measurable.

(1) We consider $F = L(R, F)$ and taking $Z = F'$, we obtain an F''-valued density satisfying

$$\langle B_t, x'\rangle = \int_{[0,t]} \langle H_s, x'\rangle \, dA_s \quad \text{for } x' \in F'.$$

If F is separable then

$$|B|_t = \int_{[0,t]} ||H_s|| \, dA_s.$$

(2) If F is the dual of a Banach space Z, then H is F-valued and

$$\langle z, B_t\rangle = \int_{[0,t]} \langle z, H_s\rangle \, dA_s \quad \text{for } z \in Z.$$

If Z is separable then $\langle z, H\rangle$ is $\mathcal{M}$-measurable for every $z \in Z$ and

$$|B|_t = \int_{[0,t]} ||H_s|| \, dA_s.$$

(3) If F has the Radon-Nikodým property, then H is F-valued, is locally $\mu_{|A|}$-integrable and

$$B_t = \int_{[0,t]} H_s \, dA_s.$$

If, in addition, F is separable, then H is $\mathcal{M}$-measurable.

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