PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

EXPANSION OF DISCRETE AND CLOSURE-PRESERVING FAMILIES

TAKEMI MIZOKAMI

(Communicated by Dennis Burke)

ABSTRACT. In this paper, we define the classes of d-IP-expandable spaces and IP-expandable spaces, and study their properties and relations with orthocompact spaces and nonarchimedean quasi-metrizable spaces.

1. Introduction. Following [1], a space X is called CP-*expandable* if for each closure-preserving family $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X and for each family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ, there exists a closure-preserving family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset \overline{V}_\lambda \subset U_\lambda$ for each λ. In this paper, we introduce the classes of IP-*expandable* spaces and d-IP-*expandable* spaces by replacing "closure-preserving" with other conditions. Our main purpose is to study the properties of these classes and the relations with orthocompact spaces and nonarchimedean quasi-metrizable spaces.

All spaces are assumed to be T_1 topological spaces and $\mathbf{N}$ always denotes the set of natural numbers.

2. D-IP-expandability and IP-expandability. We state the definitions of d-IP-expandability and IP-expandability.

DEFINITION 2.1. We call a space X d-IP-*expandable* if for a discrete family $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X and a family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ, there exists an interior-preserving family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each λ.

DEFINITION 2.2. We call a space X IP-*expandable* if for a closure-preserving family $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X and a family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ, there exists a family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each λ and $\{V_\lambda : \lambda \in \Lambda\}$ is interior-preserving in X.

In either case, we call $\mathcal{V}$ the IP-*expansion* of $\mathcal{F}$ with respect to $\mathcal{U}$ in X. A space X is called (σ-)*orthocompact* if every open cover of X has a (σ-)interior-preserving open refinement.

PROPOSITION 2.3. *If a space X is collectionwise normal, then X is d-IP-expandable.*

PROPOSITION 2.4. *If a space X is orthocompact, then X is d-IP-expandable.*

Received by the editors December 26, 1985 and, in revised form, August 25, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54D20.
Key words and phrases. Interior-preserving, orthocompact, nonarchimedean quasi-metrizable, d-IP-expandable.

PROOF. Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.1. Assume $F_\lambda \cap U_{\lambda'} = \emptyset$ if $\lambda \neq \lambda'$. Since X is orthocompact, there exists an interior-preserving open refinement $\mathcal{W}$ of an open cover $\mathcal{U} \cup \{X - \bigcup \mathcal{F}\}$. Setting $V_\lambda = S(F_\lambda, \mathcal{W})$ for each λ, we have the IP-expansion $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{F}$ with respect to $\mathcal{U}$.

In [**5**], Michael constructed a normal, noncollectionwise normal, metacompact space X. By Proposition 2.4, X is d-IP-expandable. Thus, the converse of Proposition 2.3 is not true. Also, Scott constructed a countably compact space X which is not orthocompact [**8**, Example 4.5]. Obviously, X is d-IP-expandable. Hence the converse of Proposition 2.4 is also not true. The following gives a simple sufficient condition for a d-IP-expandable space to be orthocompact.

THEOREM 2.5. *If a space X is submetacompact and d-IP-expandable, then X is orthocompact.*

PROOF. Let $\mathcal{U}$ be an open cover of X. Since X is submetacompact, that is θ-refinable, by [**11**] there exists an open refinement $\bigcup_{n=1}^\infty \mathcal{U}_n$ of $\mathcal{U}$ and a closed cover $\{F_n : n \in \mathbf{N}\}$ of X such that for each n, $\mathcal{U}_n$ covers F_n and $\mathcal{U}_n$ is point-finite at each point of F_n. For each $n, k \in \mathbf{N}$, set the closed set by

$$E_{nk} = \{x \in F_n : \operatorname{ord}(x, \mathcal{U}_n) \leq k\},$$

where $\operatorname{ord}(x, \mathcal{U}_n) = |\{U \in \mathcal{U}_n : x \in U\}|$. Then $\{E_{nk} : n, k \in \mathbf{N}\}$ satisfies the following conditions

(1) For each n, k, $\bigcup_{k=1}^\infty E_{nk} = F_n$ and $E_{nk} \subset E_{nk+1}$.

(2) For each n, E_{n1} is the union of a discrete family $\mathcal{E}_{n1}$ of closed subsets of F_n such that each $E \in \mathcal{E}_{n1}$ is contained in some $U(E) \in \mathcal{U}_n$.

(3) For each n and each $k \geq 2$, if T is a closed subset of E_{nk} such that $T \cap E_{nk-1} = \emptyset$, then T is the union of a discrete family $\mathcal{E}(T)$ of closed subsets of E_{nk} such that each $E \in \mathcal{E}(T)$ is contained in some $U(E) \in \mathcal{U}_n$.

Let $n \in \mathbf{N}$ be fixed for a while. Since X is d-IP-expandable, there exists the IP-expansion $\mathcal{V}_{n1}$ of $\mathcal{E}_{n1}$ with respect to $\{U(E) : E \in \mathcal{E}_{n1}\}$. By (3) and by d-IP-expandability of X again, there exists the IP-expansion $\mathcal{V}_{n2}$ of $\mathcal{E}(E_{n2} - \bigcup \mathcal{V}_{n1})$ with respect to $\{U(E) : E \in \mathcal{E}(E_{n2} - \bigcup \mathcal{V}_{n1})$. Repeating this process, we can get a sequence $\{\mathcal{V}_{nk} : k \in \mathbf{N}\}$ of IP-expansions. It is easy to see that $\bigcup\{\mathcal{V}_{nk} : n, k \in \mathbf{N}\}$ is a σ-interior-preserving open refinement of $\mathcal{U}$. By [**3**], X is countably metacompact. Let $\{V_{nk} : n, k \in \mathbf{N}\}$ be a point-finite open refinement of $\{\bigcup \mathcal{V}_{nk} : n, k \in \mathbf{N}\}$ such that $V_{nk} \subset \bigcup \mathcal{V}_{nk}$ for each n, k. It is easy to see that

$$\bigcup [\{V_{nk} \cap V : V \in \mathcal{V}_{nk}\} : n, k \in \mathbf{N}]$$

is an interior-preserving open refinement of $\mathcal{U}$.

The converse of Theorem 2.5 is not true, because there is a noncountably metacompact orthocompact space [**8**, Example 4.2]. A quasi-metric d on a set X with the property that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for each $x, y, z \in X$, is called *nonarchimedean*, and the space (X, d) is called *nonarchimedean quasi-metrizable*. As is well known, nonarchimedean quasi-metrizable spaces are characterized as spaces that have a σ-interior-preserving base.

COROLLARY 2.6. *Let X be a developable space. Then X is d-IP-expandable if and only if X is nonarchimedean quasi-metrizable.*

PROOF. Both if and only if parts follow easily from Theorem 2.5 and [**4**, Theorem 14].

It follows directly from the preceding result that every nonarchimedean quasi-metrizable space is σ-orthocompact, but the converse is not true [**4**, p. 116]. On the other hand, Sorgenfrey lines show that nonarchimedean quasi-metrizable spaces need not be developable. The following is not known:

QUESTION 2.7. If a space X is developable and quasi-metrizable, then is X d-IP-expandable?

This is equivalent to the well-known problem, due to Junnila, whether every developable quasi-metrizable space is nonarchimedean quasi-metrizable.

THEOREM 2.8. *For a space X, the following are equivalent:*

(1) *X is an orthocompact developable space.*

(2) *X has a development $\{\mathcal{U}_n : n \in \mathbf{N}\}$ such that each $\mathcal{U}_n$ is interior-preserving in X.*

(3) *X is a d-IP-expandable developable space.*

(4) *X is a semistratifiable, nonarchimedean quasi-metrizable space.*

PROOF. (1)$\to$(2) is trivial. (2)$\to$(3): Under (2), X is a submetacompact σ-orthocompact space. Then X is orthocompact. By Proposition 2.4, X is d-IP-expandable. (3)$\to$(4) follows from Corollary 2.6. (4)$\to$(1): Under (4), X is a submetacompact σ-orthocompact space, and therefore X is orthocompact. Since a semistratifiable γ-space is developable [**7**], X is developable.

COROLLARY 2.9. *If for each $n \in \mathbf{N}$, X_n is an orthocompact developable space, then so is $\prod_{n=1}^{\infty} X_n$.*

PROOF. This follows from the fact that semistratifiability and having a σ-interior-preserving base are countably productive properties.

A space X is said to have *property* (P) provided that for a closed G_δ-set F of X, there exists a family $\mathcal{U}$ of open subsets of X satisfying the following:

(1) $\mathcal{U}/(X - F)$ is interior-preserving in $X - F$.

(2) For each open subset V of X, there exists $U \in \mathcal{U}$ such that $V \cap F = U \cap F \subset U \subset V$.

THEOREM 2.10. *If a space X is nonarchimedean quasi-metrizable, then X has the property* (P).

PROOF. Write $F = \bigcap_{n=1}^{\infty} O_n$, where for each n O_n is open in X and $O_{n+1} \subset O_n$. Let $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a base for X, where for each n $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ and $\mathcal{B}_n$ is interior-preserving in X. Let $\{\mathcal{B}(\lambda) : \lambda \in \Lambda\}$ be the totality of subfamilies of $\bigcup_{n=1}^{\infty}(\mathcal{B}_n/O_n)$. Then it is easy to see that $\mathcal{U} = \{\bigcup \mathcal{B}(\lambda) : \lambda \in \Lambda\}$ is the desired family.

COROLLARY 2.11. *If a space X is perfect and nonarchimedean quasi-metrizable then X is d-IP-expandable.*

PROOF. Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.1. We apply the theorem to the closed subset $F = \bigcup\{F_\lambda : \lambda \in \Lambda\}$ to get a family $\mathcal{W}$ of open subsets of X satisfying (1) and (2) above with $\mathcal{U}$ replaced by

$\mathcal{W}$. Observe that for each $F_\lambda \cup (U_\lambda - F) = U_{\lambda'}$ is open in X such that $F_\lambda = U_{\lambda'} \cap F$. For each λ, take $W_\lambda \in \mathcal{W}$ such that

$$W_\lambda \cap F = F_\lambda \subset W_\lambda \subset U_{\lambda'}.$$

Then it is easy to see that $\{W_\lambda : \lambda \in \Lambda\}$ is the IP-expansion of $\mathcal{F}$ with respect to $\mathcal{U}$.

We call a family $\mathcal{U}$ of open subsets of X an *outer base of a subset F in X* if for each open subset O with $F \subset O$ there exists $U \in \mathcal{U}$ such that $F \subset U \subset O$.

COROLLARY 2.12. *If X is perfect and nonarchimedean quasi-metrizable, then every closed subset F of X has an outer base $\mathcal{U}$ in X such that $\mathcal{U}$ is interior-preserving in $X - F$.*

THEOREM 2.13. *Let X be a developable space. Then X is IP-expandable if and only if X is d-IP-expandable.*

PROOF. The "only if" part is trivial. "If" part: Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.2. Since X is semistatifiable, by the method of [**10**], we can get a family $\mathcal{H} = \bigcup_{n=1}^\infty \mathcal{H}_n$ of closed subsets of X such that each $\mathcal{H}_n$ is discrete in X and for each λ, there exists $\mathcal{H}(\lambda) \subset \mathcal{H}$ such that $F_\lambda = \bigcup \mathcal{H}(\lambda)$. Write $\mathcal{H}(\lambda) = \bigcup_{n=1}^\infty \mathcal{H}(\lambda, n)$, where $\mathcal{H}(\lambda, n) = \mathcal{H}(\lambda) \cap \mathcal{H}_n$ for each n. By Theorem 2.8, X is nonarchimedean quasi-metrizable. Therefore, by Corollary 2.10, each $H \in \mathcal{H}$ has an outer base $\mathcal{U}(H)$ in X such that $\mathcal{U}(H)$ is interior-preserving in $X - H$. For each $\lambda \in \Lambda$ and each $H \in \mathcal{H}(\lambda, n)$, $n \in \mathbf{N}$, we choose $U(H) \in \mathcal{U}(H)$ such that $U(H) \subset U_\lambda \cap O_n(F_\lambda)$. ($\{O_n(F) : n \in \mathbf{N}\}$ is the semistratification of F in X.) Set

$$W_\lambda = \bigcup \{U(H) : H \in \mathcal{H}(\lambda)\}.$$

Then it is easy to see that $F_\lambda \subset W_\lambda \subset U_\lambda$ for each λ. To see that $\{W_\lambda - F_\lambda : \lambda \in \Lambda\}$ is interior-preserving in X, let $p \in \bigcap \{W_\lambda - F_\lambda : \lambda \in \Lambda_0\}$ for $\Lambda_0 \subset \Lambda$. There exists $n \in \mathbf{N}$ such that $p \in X - O_n(\bigcup \{F_\lambda : \lambda \in \Lambda_0\})$. Since

$$\bigcup_{k=1}^{n-1} \left[U(H) : H \in \bigcup \{\mathcal{H}(\lambda, k) : \lambda \in \Lambda_0\} \right]$$

is interior-preserving at p, we obtain an open set O of X such that $p \in O \subset \bigcap \{W_\lambda - F_\lambda : \lambda \in \Lambda_0\}$.

Nagami introduced the class of L-spaces, which lies between the classes of Lašnev spaces and M_1-spaces [**6**]. He called a space X an *L-space* if X is a paracompact σ-space such that each closed subset F of X has a closure-preserving outer base and at the same time has an outer base which is interior-preserving in $X - F$. From the definition, we easily have the following result.

THEOREM 2.14. *Let X be a stratifiable space. Then X is an L-space if and only if X is IP-expandable.*

There exists a stratifiable space X which is not an L-space [**6**, Example 2.2]. Therefore, d-IP-expandability need not imply IP-expandability even if X is orthocompact.

Following [**2**], a space X is called *D-expandable* if for any discrete family $\{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X and each family $\{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$ for each λ and $F_\lambda \cap U_\mu = \varnothing$ whenever $\lambda \neq \mu$, there exists a dissectable family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each λ. (For the definition of dissectable families, refer to [**2**].) Brandenburg showed that a space is D-paracompact if and only if it is submetacompact and D-expandable [**2**, Theorem 1].

THEOREM 2.15. *If a space X is semistratifiable, then d-IP-expandability implies D-expandability.*

PROOF. It suffices to show that every interior-preserving family $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of open subsets of a semistratifiable space X is dissectable. Since $\{X - U_\lambda : \lambda \in \Lambda\}$ is a closure-preserving family of closed subsets of X, by the method of [**10**] there exists a family $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of closed subsets of X satisfying the following:

(1) Each $\mathcal{H}_n$ is discrete in X.

(2) For each subset $\Lambda_0 \subset \Lambda$, if $p \in \bigcap\{U_\lambda : \lambda \in \Lambda_0\}$ then $p \in H \subset \bigcap\{U_\lambda : \lambda \in \Lambda_0\}$ for some $H \in \mathcal{H}$.

For each $n \in \mathbf{N}$ and each $\lambda \in \Lambda$, set

$$H_{\lambda n} = \bigcup\{H \in H_n : H \subset U_\lambda\}.$$

Then by (2), $U_\lambda = \bigcup_{n=1}^{\infty} H_{\lambda n}$ for each λ. Since $\mathcal{U}$ is interior-preserving in X, it is easy to see that $\mathcal{U}$ is dissectable in X.

However, these notions of expandability are very different, because there exists a nonorthocompact developable space X (for example, $X = \langle H_0, \mathcal{U}\rangle$ in [**9**, Example 4.9]). Therefore, D-expandability need not imply d-IP-expandability. Also, there exists a perfect subparacompact nonarchimedean quasi-metrizable space [**2**, Example 1], which is not D-paracompact. Therefore, the converse is also not true.

REFERENCES

1. C. R. Borges, *Expansion of closure-preserving collections and metrizability*, Math. Japon. **28** (1983), 67–71.
2. H. Brandenburg, *On D-paracompact spaces*, Topology Appl. **20** (1985), 17–27.
3. R. F. Gittings, *Some remarks on weak covering conditions*, Canad. J. Math. **26** (1974), 1152–1156.
4. J. Kofner, *On quasi-metrizability*, Topology Proc. **5** (1980), 111–138.
5. E. A. Michael, *Point-finite and locally finite coverings*, Canad. J. Math. **7** (1955), 275–279.
6. K. Nagami, *The equality of dimensions*, Fund. Math. **106** (1980), 239–246.
7. S. I. Nedev and M. M. Čoban, *On the theory of o-metrizable spaces*. III, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **27** (1972), 10–15.
8. B. M. Scott, *Toward a product theory of orthocompactness*, Studies in Topology, Academic Press, 1975, pp. 517–537.
9. ——, *More about orthocompactness*, Topology Proc. **5** (1980), 155–184.
10. S. Siwiec and J. Nagata, *A note on nets and metrization*, Proc. Japan Acad. **44** (1968), 623–627.
11. J. M. Worrel, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. **17** (1965), 820–830.

DEPARTMENT OF MATHEMATICS, JOETSU UNIVERSITY OF EDUCATION, JOETSU, NIIGATA 943, JAPAN