

SPANS OF SIMPLE TRIODS

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ABSTRACT. We calculate and estimate the spans of objects in certain classes of continua; our main interest is in those objects with positive span.

In 1964, the concept of the span of a metric space was introduced by A. Lelek (see [2, p. 209]). Since that time, some modified versions of the span were considered (cf. [4 and 5]). To date the metric spaces for which the various spans have been explicitly calculated have mainly been objects for which the span is zero. It is still an unsettled problem whether or not continua of span zero are arc-like (see [3, p. 93]). In this paper, we calculate and estimate the spans of objects in certain classes of continua, and our main interest is in those objects with positive span.

Questions have been raised concerning the relationships between these various versions of spans. For instance, it was conjectured that the surjective span is always at least a half of the span (see [5, p. 37]). We answer this conjecture in the affirmative for some types of simple triods (see 2.5). Another problem concerning simple triods asks if the semispan and the span are always equal (see [5, p. 38]). This problem is solved positively in §2 for some classes of simple triods.

In 1961, the width of a tree-like continuum was introduced by C. E. Burgess (see [1, p. 447]). The relationship of the width and the span is only vaguely known (cf. [6, p. 191]). We involve, in §2, the widths of simple triods in our estimations of the span (see 2.4, for example).

1. Preliminaries. All spaces studied in this paper are assumed to be nonempty metric spaces. The distance between two points x, y of a metric space X is denoted by $d(x, y)$. If $x \in X$ and $A, B \subset X$ are nonempty subsets, we write

$$d(x, A) = \text{Inf}\{d(x, y) : y \in A\},$$
$$d(A, B) = \text{Inf}\{d(x, y) : x \in A, y \in B\}.$$

If $a \in X$ and $\varepsilon > 0$, the *open ball* and the *closed ball*, centered at the point a with radius ε , are defined by the formulas

$$B(a, \varepsilon) = \{x \in X : d(a, x) < \varepsilon\},$$
$$\overline{B}(a, \varepsilon) = \{x \in X : d(a, x) \leq \varepsilon\},$$

respectively. Notice that the closure $\text{cl } B(a, \varepsilon)$ is contained in $\overline{B}(a, \varepsilon)$, but the closed ball can be larger, as a set, than the closure of the open ball.

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All mappings are assumed to be continuous functions. If $f: X \rightarrow Y$ and $g: X \rightarrow Z$ are mappings, the *diagonal mapping* $f\Delta g: X \rightarrow Y \times Z$ into the product $Y \times Z$ is defined by the formula

$$(f\Delta g)(x) = (f(x), g(x)) \quad (x \in X),$$

so that $f\Delta g$ is continuous. The *standard projections* $p_1, p_2: X \times X \rightarrow X$ are mappings defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ for $(x, y) \in X \times X$.

Let X be a nonempty metric space. The *surjective span* $\sigma^*(X)$ of X is the least upper bound of real numbers α such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $d(x, y) \geq \alpha$ for $(x, y) \in C_\alpha$ and

$$(\sigma^*) \quad p_1(C_\alpha) = p_2(C_\alpha) = X.$$

Relaxing condition (σ^*) to the conditions

$$(\sigma) \quad p_1(C_\alpha) = p_2(C_\alpha),$$

$$(\sigma_0^*) \quad p_1(C_\alpha) = X,$$

$$(\sigma_0) \quad p_1(C_\alpha) \subset p_2(C_\alpha),$$

we obtain the definitions of the *span* $\sigma(X)$, the *surjective semispan* $\sigma_0^*(X)$, and the *semispan* $\sigma_0(X)$ of X , respectively.

By a *continuum* we mean a compact metric space which is connected.

1.1. *If X is a metric space and $\tau = \sigma^*, \sigma, \sigma_0^*, \sigma_0$, then $\tau(X)$ is the least upper bound of real numbers α such that there exist nonempty connected sets C and continuous mappings $f, g: C \rightarrow X$ with $d[f(c), g(c)] \geq \alpha$ for $c \in C$ and*

$$(\sigma^*)' \quad f(C) = g(C) = X,$$

$$(\sigma)' \quad f(C) = g(C),$$

$$(\sigma_0^*)' \quad g(C) = X,$$

$$(\sigma_0)' \quad f(C) \subset g(C),$$

respectively. Moreover, if X is a compact metric space, then C then be assumed to be continua.

PROOF. Observe that, given C_α , then taking $C = C_\alpha$, $f = p_1$, and $g = p_2$, condition (τ) implies condition $(\tau)'$. On the other hand, given C, f and g , then taking $C_\alpha = (f\Delta g)(C)$, condition $(\tau)'$ implies condition (τ) . Moreover, if X is a compact metric space and $C_\alpha \subset X \times X$ is a connected set satisfying (τ) , then the closure $C = \text{cl } C_\alpha$ also satisfies (τ) , as well as the same requirement for the distances $d(x, y)$, and C is a continuum.

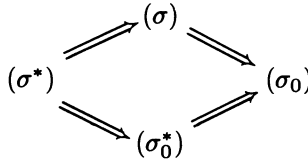
1.2. *If X is a metric space, then*

$$0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X,$$

$$0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \text{diam } X,$$

(see [4, p. 207]).

PROOF. This follows directly from the implications:



1.3. If X is a metric space and $A \subset X$ is a nonempty subset, then $\sigma(A) \leq \sigma(X)$ and $\sigma_0(A) \leq \sigma_0(X)$ (see [4, p. 207]).

1.4. If X is a metric space, C is a continuum, and $f, g: C \rightarrow X$ are mappings satisfying condition $(\tau)'$, where $\tau = \sigma^*, \sigma, \sigma_0^*, \sigma_0$, then there exists a point $c_0 \in C$ such that $d[f(c_0), g(c_0)] \leq \tau(X)$.

PROOF. Suppose on the contrary that $d[f(c), g(c)] > \tau(X)$ for each $c \in C$. Since C is compact, there is a number $\alpha_0 > \tau(X)$ such that $d[f(c), g(c)] \geq \alpha_0$ for $c \in C$. This means that α_0 is one of the numbers α appearing in the definition of $\tau(X)$, so that $\tau(X) \geq \alpha_0$, a contradiction.

By an *arc* we understand a space homeomorphic to the unit interval $[0, 1]$ of the real line.

1.5. If X is an arc, then $\sigma^*(X) = \sigma(X) = \sigma_0^* = \sigma_0(X) = 0$ (see [5, p. 36]).

By a *simple closed curve* we understand a space homeomorphic to the unit circle $\{(u, v): u^2 + v^2 = 1\}$ on the plane. A *tree* is a space homeomorphic to a connected one-dimensional polyhedron which contains no simple closed curve. A continuum X is called *tree-like* (or *arc like*) provided there exists, for each $\varepsilon > 0$, a finite open cover of X whose members have diameters smaller than ε and whose nerve, if nondegenerate, is a tree (or an arc, respectively). Finite open covers whose nerves are arcs are called *chains*. That is why arc-like continua are also called chainable. If \mathcal{C} is a collection of sets, we denote by $|\mathcal{C}|$ the union of all members of \mathcal{C} .

Let X be a nonempty compact metric space. The *width* $w(X)$ of X is the least upper bound of real numbers α which satisfy the following condition: there exists, for each $\varepsilon > 0$, a finite open cover \mathcal{C} of X such that each member of \mathcal{C} has diameter smaller than ε and

$$\text{Max}\{d(A, |\mathcal{C}'|): A \in \mathcal{C}\} \geq \alpha$$

for each chain \mathcal{C}' contained in \mathcal{C} . This definition of the width is slightly different from that of the width of a tree-like continuum, as given in [1], but it can be shown that, in the case of tree-like continua, the two definitions are equivalent (see [6, p. 184]).

1.6. If X is a tree, then

$$w(X) = \text{Inf}\{\text{Sup}\{d(x, A): x \in X\}: A \subset X, A \text{ arc}\}$$

(see [6, p. 186]).

By a *simple triod* we understand a space T which admits a decomposition $T = A_0 \cup A_1 \cup A_2$, where A_i are arcs having a common endpoint v and $A_i \cap A_{i+1} = \{v\}$ for $i = 0, 1, 2$, and the subscripts of A_j taken mod 3. Each simple triod is a tree. The arcs A_0, A_1, A_2 are called *branches* of the simple triod T , and the point v is called the *branch-point* of T .

1.7. If T is a simple triod with branches A_0, A_1, A_2 , then

$$w(T) = \text{Min}\{\text{Max}\{d(x, A_{i+1} \cup A_{i+2}) : x \in A_i\} : i = 0, 1, 2\},$$

where the subscripts are taken mod 3 (see [6, p. 187]).

1.8. If T is a simple triod, then $w(T) \leq \sigma^*(T)$ (see [4, p. 210; 7, p. 208]).

For some simple triods, the weak inequality in 1.8 becomes the equality (cf. 3.1). However, in general, this is not true. It is known that there exists, for each $\varepsilon > 0$, a simple triod T on the plane such that $\sigma(T) = 1$ and $w(T') < \varepsilon$ for each simple triod T' contained in T (see [6, p. 195]). In such a triod T , a subtrioid T' can be selected with $\sigma^*(T') = \sigma(T)$, whence $\sigma^*(T') = 1$. Thus we can have simple triods with surjective span equal 1 and the width as small as one wishes.

2. Main results. The first theorem gives an estimate for the semispan of certain continua related to simple triods.

2.1. THEOREM. If K and L are continua such that $K \cap L = \{v\}$, $L \subset \overline{B}(v, \delta)$, where $\delta \geq 0$, and $\sigma_0(L) \leq \delta + \sigma_0(K)$, then $\sigma_0(K \cup L) \leq \delta + \sigma_0(K)$.

PROOF. Let $h: K \cup L \rightarrow K$ be the mapping defined by

$$h(x) = \begin{cases} x & \text{if } x \in K, \\ v & \text{if } x \in L. \end{cases}$$

Let $f, g: C \rightarrow K \cup L$ be mappings such that C is a continuum and $f(C) \subset g(C)$. Considering the composites $hf, hg: C \rightarrow K$ we see, by 1.4, that there exists a point $c \in C$ such that $d[hf(c), hg(c)] \leq \sigma_0(K)$. We distinguish six cases as follows.

Case 1. $f(c) \in K, g(c) \in K$. Then $hf(c) = f(c)$ and $hg(c) = g(c)$, whence

$$d[f(c), g(c)] = d[hf(c), hg(c)] \leq \sigma_0(K) \leq \delta + \sigma_0(K).$$

Case 2. $f(c) \in K, g(c) \in L$. Then $hf(c) = f(c)$ and $hg(c) = v$, whence

$$\begin{aligned} d[f(c), g(c)] &\leq d[f(c), v] + d[v, g(c)] \leq d[f(c), v] + \delta \\ &= d[hf(c), hg(c)] + \delta \leq \sigma_0(K) + \delta. \end{aligned}$$

Case 3. $f(c) \in L, g(c) \in K$. Then $hf(c) = v$ and $hg(c) = g(c)$, whence

$$\begin{aligned} d[f(c), g(c)] &\leq d[f(c), v] + d[v, g(c)] \leq \delta + d[v, g(c)] \\ &= \delta + d[hf(c), hg(c)] \leq \delta + \sigma_0(K). \end{aligned}$$

Case 4. $f(c) \in L, g(c) \in L$ and

$$(1) \quad (f\Delta g)(C) \cap [(L \times \{v\}) \cup (\{v\} \times L)] = \phi.$$

In this case, we observe that the sets A and B defined by the formulas

$$\begin{aligned} A &= (L \setminus \{v\}) \times (L \setminus \{v\}), \\ B &= [(K \cup L) \times (K \setminus \{v\})] \cup [(K \setminus \{v\}) \times (K \cup L)] \end{aligned}$$

are disjoint open subsets of the product $(K \cup L) \times (K \cup L)$ and

$$(f\Delta g)(C) \subset [(K \cup L) \times (K \cup L)] \setminus [(L \times \{v\}) \cup (\{v\} \times L)] = A \cup B,$$

by (1). Also, the point $(f(c), g(c))$ belongs to $(f\Delta g)(C)$, and $f(c), g(c) \in L \setminus \{v\}$, by (1). Thus $(f(c), g(c)) \in A$ which, by the connectedness of $(f\Delta g)(C)$, implies

that $(f\Delta g)(C) \subset A \subset L \times L$. Consequently, $f(C) \subset g(C) \subset L$, and, by 1.4, there exists a point $c' \in C$ such that

$$d[f(c'), g(c')] \leq \sigma_0(L) \leq \delta + \sigma_0(K).$$

Case 5. $(f\Delta g)(C) \cap (L \times \{v\}) \neq \emptyset$. Then there exist a point $c'' \in C$ with $f(c'') \in L$ and $g(c'') = v$, whence

$$d[f(c''), g(c'')] = d[f(c'')v] \leq \delta \leq \delta + \sigma_0(K).$$

Case 6. $(f\Delta g)(C) \cap (\{v\} \times L) \neq \emptyset$. Then there exists a point $c''' \in C$ with $f(c''') = v$ and $g(c''') \in L$, whence

$$d[f(c'''), g(c''')] = d[v, g(c''')] \leq \delta \leq \delta + \sigma_0(K).$$

In all six cases, we have established the existence of a point $c \in C$ such that the distance $d[f(c), g(c)]$ is not greater than $\delta + \sigma_0(K)$. If all these distances are supposed to be not less than a number $\alpha \geq 0$, then $\alpha \leq \delta + \sigma_0(K)$. The semispan $\sigma_0(K \cup L)$ is, by 1.1, the least upper bound of such α 's, and therefore

$$\sigma_0(K \cup L) \leq \delta + \sigma_0(K).$$

2.2. THEOREM. *If K and L are continua such that $K \cap L = \{v\}$, $L \subset \overline{B}(v, \delta)$, where $\delta \geq 0$, and $\sigma(L) \leq \delta + \sigma(K)$, then $\sigma(K \cup L) \leq \delta + \sigma(K)$.*

PROOF. The proof of 2.2 is a replica of that of 2.1. Everywhere in the proof of 2.1 the semispan σ_0 is to be replaced by the span σ , and the inclusion $f(C) \subset g(C)$ is to be replaced by the equality $f(C) = g(C)$.

2.3. THEOREM. *If T is a simple triod with a branch-point v and $T' \subset T$ is a simple triod such that one branch of T is contained in $\overline{B}[v, w(T')]$, then*

- (i) $\sigma(T) = \sigma_0(T) = \sigma(T') = \sigma_0(T') = w(T')$,
- (ii) $\sigma^*(T') = \sigma_0^*(T') = w(T')$.

PROOF. Let L_0, L_1, L_2 be the branches of T with $L_0 \subset \overline{B}[v, w(T')]$. Since L_0 and $L_1 \cup L_2$ are arcs, we have $\sigma_0(L_0) = \sigma_0(L_1 \cup L_2) = 0$, by 1.5. Applying 2.1 to $K = L_1 \cup L_2$, $L = L_0$ and $\delta = w(T')$, we obtain $\sigma_0(T) \leq w(T')$. Note that $w(T') \leq \sigma^*(T') \leq \sigma(T')$, by 1.2 and 1.8. Also, $T' \subset T$ implies that $\sigma(T') \leq \sigma(T)$ and $\sigma_0(T') \leq \sigma_0(T)$, by 1.3. Thus

$$\begin{aligned} w(T') &\leq \sigma(T') \leq \sigma(T) \leq \sigma_0(T) \leq w(T'), \\ w(T') &\leq \sigma(T') \leq \sigma_0(T') \leq \sigma_0(T) \leq w(T') \end{aligned}$$

which yields (i). Hence $w(T') \leq \sigma^*(T') \leq \sigma(T') = w(T')$ and

$$w(T') \leq \sigma^*(T') \leq \sigma_0^*(T') \leq \sigma_0(T') = w(T'),$$

by 1.2. Consequently, (ii) is proved.

The next theorem gives an estimate for the surjective span of a simple triod in terms of the widths of its subtriods. Note that the width of a subtriod may be greater than the width of the triod (see 3.2).

2.4. THEOREM. *If $T' \subset T$ are simple triods, then $\sigma^*(T) \geq w(T')/2$.*

PROOF. Let $p_1, p_2: T \times T \rightarrow T$ denote the standard projections of the product, that is, $p_1(x, y) = x$ and $p_2(x, y) = y$ for $(x, y) \in T \times T$. Let $\Phi: T \times T \rightarrow T \times T$ denote the coordinate-switch homeomorphism, that is, $\Phi(x, y) = (y, x)$ for $(x, y) \in T \times T$. We have $p_2 = p_1\Phi$, and the composite $\Phi\Phi$ is the identity mapping of $T \times T$. Let

$$T = L_0 \cup L_1 \cup L_2, \quad T' = L'_0 \cup L'_1 \cup L'_2$$

be the decompositions of the triods T and T' into branches $L'_i \subset L_i$ ($i = 0, 1, 2$). By 1.7, there exist points $a_i \in L'_i$ such that

$$d(a_i, L'_{i+1} \cup L'_{i+2}) \geq w(T') \quad (i = 0, 1, 2),$$

where the subscripts are taken mod 3. Let v denote the branch-point of T (and also of T'). Let M_i be the subarc of the arc L'_i having end-points v and a_i ($i = 0, 1, 2$). Hence

$$(2) \quad d(a_i, M_{i+1} \cup M_{i+2}) \geq w(T') \quad (i = 0, 1, 2).$$

Let C be the subset of the product $T \times T$ defined by the formula

$$\begin{aligned} C = & [\{a_0\} \times (M_1 \cup M_2)] \cup [(M_1 \cup M_2) \times \{a_0\}] \\ & \cup [\{a_1\} \times (M_0 \cup M_2)] \cup [(M_0 \cup M_2) \times \{a_1\}] \\ & \cup [\{a_2\} \times (M_0 \cup M_1)] \cup [(M_0 \cup M_1) \times \{a_2\}], \end{aligned}$$

so that C is the union of six connected sets. The first set meets both the fourth and the sixth sets. The second set meets the third and fifth sets. Also, the third and sixth sets meet. Therefore C is connected. Let $M = M_0 \cup M_1 \cup M_2$. It follows from the definition of C and from (2) that

$$(3) \quad p_1(C) = p_2(C) = M, \quad \Phi(C) = C,$$

$$(4) \quad d(x, y) \geq w(T') \quad \text{for } (x, y) \in C.$$

Let ε be an arbitrary number with $0 < \varepsilon < w(T')/2$. Let L be a branch of T , that is, $L = L_i$ ($i = 0, 1, 2$). We shall construct a connected subset $C(L)$ of the product $T \times T$ such that

$$(5) \quad C \cap C(L) \neq \emptyset,$$

$$(6) \quad L \setminus M \subset p_1[C(L)],$$

$$(7) \quad d(x, y) > w(T')/2 - \varepsilon \quad \text{for } (x, y) \in C(L).$$

Consider the closed balls B_0, B_1 and B_2 in the simple triod T defined by the formulas

$$B_j = \bar{B}[a_j, w(T')/2 - \varepsilon] \quad (j = 0, 1, 2).$$

According to (2), the centers of these balls are of distance greater than or equal to $w(T')$ from each other. Thus these three balls are pairwise disjoint closed subsets of T , and there exists a number $\eta > 0$ such that if $x, y \in T$ and the points x, y belong to two different balls B_0, B_1, B_2 , then $d(x, y) > \eta$. Let b_i denote the endpoint of the arc $L = L_i$ different from v , so that $a_i \in L$ and b_i may coincide with a_i , but not necessarily. We consider the arc L as being ordered from its endpoint v to the endpoint b_i . If $x, y \in L$, we denote by $[x, y]$ the subarc of L with endpoints x and

y. We do not exclude degenerate subarcs, that is, the singletons $[x, x] = \{x\}$. Let us define a sequence x_1, x_2, \dots of points of L as follows. Let $x_1 = a_i$. Therefore $x_1 \in B_i$. Suppose x_k is defined ($k \geq 1$) such that $x_k \in B_{j_k}$, where $j_k = 0, 1, 2$. We let x_{k+1} be the first point of $[x_k, b_i]$ belonging to the union of two other balls, that is, belonging to $B_{j_{k+1}} \cup B_{j_{k+2}}$ with subscripts taken mod 3, if such points in $[x_k, b_i]$ exist. If they do not, x_{k+1} is not defined and the sequence stops. Observe that if x_{k+1} is defined, then the arc $[x_k, x_{k+1}]$ has endpoints in two different balls B_0, B_1, B_2 , whence its diameter is greater than η . The arc L cannot contain an infinite chain of such arcs, and so, indeed, the sequence comes to a stop. In this way, we have defined an ordered finite sequence of points x_1, x_2, \dots, x_n of L . We have $x_k \in B_{j_k}$ for $k = 1, 2, \dots, n$, and $j_k \neq j_{k+1}$ for $k = 1, 2, \dots, n-1$. Also, $j_1 = i$ and the last point x_n may, but not necessarily, coincide with the endpoint b_i of L .

The arc $[x_k, x_{k+1}]$ has endpoints in B_{j_k} and $B_{j_{k+1}}$, respectively. Moreover, by the way it was selected, $[x_k, x_{k+1}]$ is disjoint with the third remaining ball $B_{3-(j_k+j_{k+1})}$. We now define two sequences D_1, \dots, D_n and E_1, \dots, E_n of connected subsets of $T \times T$ by the formulas

$$\begin{aligned} D_k &= [x_k, x_{k+1}] \times \{a_{3-(j_k+j_{k+1})}\} & (k = 1, \dots, n-1), \\ D_n &= [x_n, b_i] \times \{a_{j_{n+1}}\}, \\ E_k &= \{x_k\} \times (M_{j_{k+1}} \cup M_{j_{k+2}}) & (k = 1, \dots, n), \end{aligned}$$

where the subscripts are taken mod 3. The set $C(L)$ is defined to be the union

$$C(L) = (D_1 \cup \dots \cup D_n) \cup (E_1 \cup \dots \cup E_n).$$

Let $l = 3 - (j_k + j_{k+1})$, that is, l is the number among 0, 1, 2 which is different from both j_k and j_{k+1} . Hence $l = j_k + 1 \pmod{3}$ or $l = j_k + 2 \pmod{3}$, and also $l = j_{k+1} + 1 \pmod{3}$ or $l = j_{k+1} + 2 \pmod{3}$. We see that the point (x_k, a_l) belongs to both sets D_k and E_k , and the point (x_{k+1}, a_l) belongs to both sets D_k and E_{k+1} ($k = 1, \dots, n-1$). Furthermore, $(x_n, a_{j_{n+1}})$ belongs to both D_n and E_n . Hence $D_k \cap E_k \neq \emptyset$ for $k = 1, \dots, n$ and $D_k \cap E_{k+1} \neq \emptyset$ for $k = 1, \dots, n-1$. It follows that the set $C(L)$ is connected. Since $x_1 = a_i$, the set E_1 is $\{a_i\} \times (M_{j_{k+1}} \cup M_{j_{k+2}})$, and so it is either the first or the third or the fifth set appearing in the definition of C , which implies (5). Inclusion (6) is also satisfied because

$$\begin{aligned} L \setminus M \subset [a_i, b_i] &= [x_1, b_i] = \bigcup_{k=1}^{n-1} [x_k, x_{k+1}] \cup [x_n, b_i] \\ &= \bigcup_{k=1}^{n-1} p_1(D_k) \cup p_1(D_n) = p_1 \bigcup_{k=1}^n D_k \subset p_1[C(L)]. \end{aligned}$$

To prove (7), let us take a point $(x, y) \in C(L)$. If $(x, y) \in D_k$ ($k = 1, \dots, n-1$), then $x \in [x_k, x_{k+1}]$ and $y = a_l$. We have observed that the arc $[x_k, x_{k+1}]$ is disjoint with the closed ball B_l which is centered at a_l . Thus

$$d(x, y) = d(x, a_l) > w(T')/2 - \varepsilon.$$

If $(x, y) \in D_n$, then $x \in [x_n, b_i]$ and $y = a_{j_{n+1}}$. But $x_n \in B_{j_n}$, and x_{n+1} does not exist meaning that the arc $[x_n, b_i]$ is disjoint with $B_{j_{n+1}} \cup B_{j_{n+2}}$; hence,

in particular, it is disjoint with the closed ball B_{j_n+1} which is centered at a_{j_n+1} . Thus

$$d(x, y) = d(x, a_{j_n+1}) > w(T')/2 - \varepsilon.$$

If $(x, y) \in E_k$ ($k = 1, \dots, n$), then $x = x_k$ and $y \in M_{j_k+1} \cup M_{j_k+2}$. But $x_k \in B_{j_k}$, where the closed ball B_{j_k} is centered at a_{j_k} , and we use (2) to infer that

$$\begin{aligned} d(x, y) &= d(x_k, y) \geq d(a_{j_k}, y) - d(a_{j_k}, x_k) \\ &\geq w(T') - [w(T')/2 - \varepsilon] \\ &= w(T')/2 + \varepsilon > w(T')/2 - \varepsilon. \end{aligned}$$

This concludes the proof of (7). To end the proof of the theorem, consider the set

$$F = C \cup \bigcup_{L=L_0, L_1, L_2} C(L).$$

Since C and $C(L)$ are connected, it follows from (5) that F is connected. Moreover, we have

$$\begin{aligned} T &= L_0 \cup L_1 \cup L_2 = M \cup \bigcup_{L=L_0, L_1, L_2} (L \setminus M) \\ &\subset p_1(C) \cup \bigcup_{L=L_0, L_1, L_2} p_1[C(L)] = p_1(F), \end{aligned}$$

by (3) and (6). By (4) and (7), we obtain the inequality $d(x, y) > w(T')/2 - \varepsilon$ for each $(x, y) \in F$. Also $\emptyset \neq C \subset F$ and $C = \Phi(C) \subset \Phi(F)$, by (3). Consequently, the set $G = F \cup \Phi(F)$ is a connected subset of $T \times T$ and

$$\begin{aligned} T &\subset p_1(F) \subset p_1(G) \subset T, \\ T &\subset p_1(F) = p_1\Phi\Phi(F) = p_2\Phi(F) \subset p_2(G) \subset T, \end{aligned}$$

which yields $p_1(G) = p_2(G) = T$. Finally, the inequality $d(x, y) > w(T')/2 - \varepsilon$ is clearly true for each $(x, y) \in G$ as well. The surjective span $\sigma^*(T)$ must then be not less than $w(T')/2 - \varepsilon$, and since ε was allowed to be an arbitrarily small positive number, the conclusion of 2.4 follows.

2.5. COROLLARY. *If T is a simple triod with a branch-point v and $T' \subset T$ is a simple triod such that one branch of T is contained in $\overline{B}[v, w(T')]$, then $\sigma(T) \leq 2\sigma^*(T)$.*

To get 2.5, apply 2.3 and 2.4. The simplest example of a triod shows that the weak inequalities in both 2.4 and 2.5 cannot be replaced by the equalities (see 3.1). Combining 2.3 and 2.4, we obtain estimates for the values of all spans of T with the exception of $\sigma_0^*(T)$.

3. Examples. We give some examples of simple triods in the Euclidean 2-space \mathcal{R}^2 with the ordinary Pythagorean distance d .

3.1. EXAMPLE. *There exists a simple triod T in \mathcal{R}^2 such that $T \subset \overline{B}[v, w(T)]$, where v is the branch-point of T , and*

$$\sigma(T) = \sigma_0(T) = \sigma^*(T) = \sigma_0^*(T) = w(T) = 1.$$

Let $v = (0, 0)$ be the origin in \mathcal{R}^2 . For two points $a, b \in \mathcal{R}^2$, we denote by \overline{ab} the straight line interval having a and b as endpoints. We take $a_0 = (-1, 0)$, $a_1 = (0, 1)$, $a_2 = (1, 0)$, and $A_i = \overline{a_i v}$ ($i = 0, 1, 2$). Let $T = A_0 \cup A_1 \cup A_2$.

We observe that $d(a_i, A_{i+1} \cup A_{i+2}) = 1$ and $d(x, A_{i+1} \cup A_{i+2}) < 1$ for $x \in A - \{a_i\}$ and for $i = 0, 1, 2$. Applying 1.7, we see that $w(T) = 1$. From 1.8, we see that $\sigma^*(T) \geq 1$. To show the desired equality, by 1.2, we just need to show that $\sigma_0(T) \leq 1$.

Let $C \subset T \times T$ be a nonempty connected set such that $p_1(C) \subset p_2(C)$, where $p_1, p_2: T \times T \rightarrow T$ are the standard projections. Then $p_2(C)$ is a connected subset of T . The vertex v cuts the simple triod T into three components which are subsets of the three polygonal arcs forming T . If $v \notin p_2(C)$, the connected set $p_2(C)$ lies on one of the arcs and so does $p_1(C)$. The semispan of each arc is zero, by 1.5, whence C contains points (q, q') , with $d(q, q')$ arbitrarily small; in particular $d(q, q') \leq 1$ for some $(q, q') \in C$. If $v \in p_2(C)$, then there exists a point $q_0 \in T$ such that $(q_0, v) \in C$ and $d(q_0, v) \leq 1$. In both cases, there does not exist a number $\alpha > 1$ such that $d(q, q') \geq \alpha$ for $(q, q') \in C$. Consequently, $\sigma_0(T) \leq 1$.

3.2. EXAMPLE. *There exists, for each number ε with $0 < \varepsilon < 1$, a simple triod T^ε in \mathcal{R}^2 such that $T \subset T^\varepsilon$, where T is the triod from 3.1, and $w(T^\varepsilon) = \varepsilon$, while $w(T) = 1$.*

Let $T^\varepsilon = T \cup B'$, where $b = (-\varepsilon, \varepsilon)$, $c = (-\varepsilon, 1)$ and $B' = \overline{a_0 b} \cup \overline{bc}$. We see that the three branches of T^ε are $A_0 \cup B'$, A_1 , and A_2 .

To determine the value of $\text{Max}\{d(x, A_0 \cup B' \cup A_2) : x \in A_1\}$ we distinguish two cases as follows.

Case 1. $x = (0, y)$, where $0 \leq y \leq \varepsilon$.

In this case $d(x, A_0 \cup B' \cup A_2) \leq \varepsilon$, since $v \in A_0 \cup B' \cup A_2$ and $d(x, v) \leq \varepsilon$.

Case 2. $x = (0, y)$, where $\varepsilon < y \leq 1$.

In this case, $d(x, A_0 \cup B' \cup A_2) = \varepsilon$ since $(-\varepsilon, y) \in B'$, $d[x, (-\varepsilon, y)] = \varepsilon$, and for each $z \in (A_0 \cup B' \cup A_2) - \{(-\varepsilon, y)\}$, $d(x, z) > \varepsilon$.

Hence, we get that $\text{Max}\{d(x, A_0 \cup B' \cup A_2) : x \in A_1\} = \varepsilon$. Since $d(a_0, A_1 \cup A_2) = 1$ and $d(a_2, (A_0 \cup B') \cup A_1) = 1$, we see that $\text{Max}\{d(x, A_1 \cup A_2) : x \in A_0 \cup B'\} \leq 1$ and $\text{Max}\{d(x, A_0 \cup B' \cup A_1) : x \in A_2\} \geq 1$. From 1.7, we conclude that $w(T^\varepsilon) = \varepsilon$.

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