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THE SEPARABLE REPRESENTATIONS OF U(H)

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ABSTRACT. In this paper we show that the separable representation theory of $\mathrm{U}(H)$ is completely analogous to that for $\mathrm{U}(\mathbf{C}^n)$, in that every separable representation is discretely decomposable and the irreducible representations all occur in the decomposition of the mixed tensor algebra of H. This was previously shown to be true (for all representations, separable and nonseparable) for the normal subgroup $\mathrm{U}(H)_\infty$, consisting of operators which are compact perturbations of the identity, by Kirillov and Ol′shanskii. In particular we show that all nontrivial representations of the unitary Calkin group are nonseparable. The proof exploits the analogue of the following fact about the Calkin algebra: if π is a nontrivial representation of the Calkin algebra and T is a normal operator on H, then every point in the spectrum of $\pi(T)$ is an eigenvalue.

1. Introduction. Let H denote a separable infinite dimensional Hilbert space. The unitary group $\mathrm{U}(H)$ in the operator norm topology is not a simple topological group. However, modulo the scalar operators, $\mathrm{U}(H)$ has a unique closed normal subgroup, namely, $\mathrm{U}(H)_\infty$, the unitary operators of the form $g = 1 + T$, where T is a compact operator; this was proven by Kadison [**1**].

Now Kirillov and Ol′shanskii, following earlier work of I. Segal, have shown that every unitary representation of $\mathrm{U}(H)_\infty$ is a discrete sum of irreducibles, and that the irreducible representations are precisely those one obtains by decomposing the mixed tensor algebra [**11**, **2**, **3**]. In this paper we will show that every *separable* unitary representation of $\mathrm{U}(H)_\infty$ has a *unique* extension to a unitary representation of $\mathrm{U}(H)$. In particular the quotient $\mathrm{U}(H)/\mathrm{U}(H)_\infty$, which is isomorphic to the identity component of the unitary group of the Calkin C^*-algebra $\mathfrak{L}(H)/\mathfrak{L}(H)_\infty$, does not have a nontrivial separable representation (this seems to be a new result, although the analogous assertion for the Calkin algebra is elementary and well known). Consequently we see that the theory of *separable* unitary representations of $\mathrm{U}(H)$ is as elementary as the corresponding theory for $\mathrm{U}(\mathbf{C}^n)$. (Note if we consider the strong operator topology instead, then $\mathrm{U}(H)_\infty$ is dense in $\mathrm{U}(H)$, and we can remove the separable condition.) Of course this is patently false when we consider nonseparable representations, for it is known that the Calkin algebra itself has $2^{\mathfrak{c}}$ ($\mathfrak{c}$ = power of the continuum) inequivalent irreducible representations, each of which induces a distinct (uniformly continuous) irreducible representation of $\mathrm{U}(H)$.

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To prove our results we use the works cited above together with the analogue (Lemma 4) of the following fact about the Calkin algebra: if π is a nontrivial representation of the Calkin algebra and $T \in \mathfrak{L}(H)$ is normal, then λ is an eigenvalue for $\pi(T)$ whenever λ is in the essential spectrum of T. This is briefly presented in §3.

One motivation for extending the results of Kirillov and Ol′shanskii from $\mathrm{U}(H)_\infty$ to $\mathrm{U}(H)$ is the theory of gauge groups. Suppose we write $H = H_+ \oplus H_-$, where H_+ and H_- are infinite dimensional. The restricted unitary group, denoted $U_{(p)}$, consists of those unitary operators on $H = H_+ + H_-$ having the matrix form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with b, c in the Schatten p-class ($1 \le p \le \infty$). When H (H_+) is the space of (positive) spinors with values in $\mathbf{C}^n$ on a d-dimensional space X, there is then a natural homomorphism $\mathrm{Map}(X, \mathrm{U}(n)) \to \mathrm{U}_{(p)}$, where $p = d + 1$ (see [**4**]). When $X = S^1$ it is known that the positive energy representations of $\mathrm{Map}(S^1, \mathrm{U}(n))$ are restrictions of certain natural representations of $\mathrm{U}_{(2)}$. There may be a link for $d > 1$ as well, although this is more speculative (see [**4** and **5**]). At any rate it is natural to ask what can be said about the separable representation theory of $\mathrm{U}_{(p)}$.

Now we have an exact sequence

$$1 \to \mathrm{U}(H)_\infty \cap \mathrm{U}_{(p)} \to (\mathrm{U}_{(p)})_o \to \mathrm{U}(H_+)/\mathrm{U}(H_+)_\infty \times \mathrm{U}(H_-)/\mathrm{U}(H_-)_\infty \to 1,$$

where the subscript o denotes identity component. The results of this paper imply that the separable representations of the (identity component of the) restricted unitary group are identical to those for $\mathrm{U}(H)_\infty \cap \mathrm{U}_{(p)}$ (see Corollary 1). Determining these may be a tractable problem, because this group is a limit of compact groups (see [**6**]).

I thank Graeme Segal for encouraging me to think about these questions. I also thank R. Kadison for pointing out to me that one can also use a theorem of Berg to prove Theorem 3.

Notation. Throughout this paper we will abbreviate "strong operator continuous unitary representation" to "unitary representation." $\prod G_i$ denotes the weak product of groups G_i.

2. U(H) separable representations.

Let H be a separable Hilbert space. We equip $\mathrm{U}(H)$ with the operator norm topology. Our first result is

THEOREM 1. *Every unitary representation of* $\mathrm{U}(H)/\mathrm{U}(H)_\infty$ *on a separable Hilbert space is a multiple of the trivial representation.*

We will need Lemmas 1–4.

LEMMA 1. *Let* π *be a unitary representation of* $\mathrm{U}(H)$. *If* $H = V \oplus V^\perp$, *where* V *and* $V^\perp$ *are both infinite dimensional, then* $H_\pi^{\mathrm{U}(V)} \neq \{0\}$.

The proof of this is a slight modification of an argument of Ol′shanskii. Let $H_1 = V$, and write $V^\perp = \sum_2^\infty \bigoplus H_j$, where each H_j is infinite dimensional. We first note that Lemma 1.3 of [**3**] holds with $\mathrm{U}(H)$ in place of $K^0(\infty)$ (there is a misprint in the proof: the value of η should be $\eta = (1 - \frac{1}{4}(m+1)^{-2})^{1/2}$). In Lemma 1.4 of [**3**] it is not necessary to assume ρ is irreducible. If we replace $K_{n_{i-1}}(n_i)$ by $\mathrm{U}(H_i)$ in the proof of Lemma 1.4 of [**3**], we can conclude $H_\pi^{\mathrm{U}(H_j)} \neq 0$ for some j, and this implies our Lemma 1.

LEMMA 2. *Suppose π is a unitary representation of* $\mathrm{U}(H)/\mathrm{U}(H)_\infty$. *Suppose* $H = \sum_1^\infty \bigoplus H_j$, *where each* H_j *is infinite dimensional. Then*

$$\bigcap_{1\leq j<\infty} H_\pi^{\mathrm{U}(H_j)} \neq \{0\}.$$

PROOF. Let V be a closed subspace of H satisfying $V = \sum \bigoplus V \cap H_j$ and $\dim(V^\perp \cap H_j) = 1$ for all j. We then have natural inclusions

$$\mathrm{U}(H_j)/\mathrm{U}(H_j)_\infty \to \mathrm{U}(V)/\mathrm{U}(V)_\infty \to \mathrm{U}(H)/\mathrm{U}(H)_\infty$$

for all j. Hence Lemma 2 follows from Lemma 1.

LEMMA 3. *Let π be a unitary representation of* $\mathrm{U}(H)/\mathrm{U}(H)_\infty$. *Then 1 is an eigenvalue for $\pi(g)$ whenever 1 is in the essential spectrum of g.*

PROOF. Let P_ε equal the spectral projection for g corresponding to the ball of radius $\varepsilon > 0$ about 1. We then have that $g_\varepsilon \to g$ uniformly, where $g_\varepsilon = P_\varepsilon^\perp g P_\varepsilon^\perp + P_\varepsilon$. On the other hand we have each $g_\varepsilon \in \prod_1^\infty \mathrm{U}(H_j) \to \mathrm{U}(H)$ for an appropriate choice of $H_1, H_2, \ldots$ (e.g. $H_j = P_{\varepsilon_j} H \ominus P_{\varepsilon_{j-1}} H$ where $\varepsilon_1 > \varepsilon_2 > \cdots$, or $H_j =$ a space of fixed vectors for g if 1 is an eigenvalue of g with infinite multiplicity). Thus there is a nonzero vector $v_0 \in H_\pi$ such that $\pi(g_\varepsilon)v_0 = v_0$ for all $\varepsilon > 0$. By strong continuity of π we conclude 1 is an eigenvalue for $\pi(g)$. □

LEMMA 4. *Suppose π is a unitary representation of* $\mathrm{U}(H)/\mathrm{U}(H)_\infty$ *which satisfies $\pi(\lambda) = \lambda^j \cdot 1$ for all $\lambda \in \mathbf{T}$, where 1 is the identity on H_π. Then if λ is in the essential spectrum of g, λ^j is an eigenvalue of $\pi(g)$.*

This follows immediately from Lemma 3.

PROOF OF THEOREM 1. Suppose π is a separable unitary representation of $\mathrm{U}(H)/\mathrm{U}(H)_\infty$. We first observe π is trivial on the scalars, $\mathbf{T}$. For otherwise on any sector H_π where $\pi|_{\mathbf{T}}$ acts via a nontrivial character, $\pi(g)$ will have uncountably many eigenvalues whenever g has uncountable essential spectrum, by Lemma 4. Thus $\pi(\mathbf{T}) \equiv 1$.

Now let P be an orthogonal projection on H with infinite rank and corank. The representation π restricted to $\mathrm{U}(PH) \to \mathrm{U}(H)$ must also vanish on the scalars, i.e. $\pi(\lambda P + 1 - P) = 1$ for all $\lambda \in \mathbf{T}$. But $\mathrm{U}(H)/\mathbf{T}\cdot\mathrm{U}(H)_\infty$ is a simple topological group, by Kadison's result [**1**]. Thus $\pi \equiv 1$. □

We now turn to our second result.

THEOREM 2. *Let π be a separable unitary representation of* $\mathrm{U}(H)_\infty$. *Then there exists a unique extension of π to a unitary representation* $\mathrm{U}(H)$.

For the proof we fix an orthonormal basis $\{\varepsilon_j\}$ for H, and we identify $\mathbf{C}^n \cong \mathrm{span}\{\varepsilon_j : 1 \leq j \leq \eta\}$.

PROOF. Kirillov and Ol′shanskii's results show there exists an extension. So suppose π is an arbitrary extension. We can suppose $H_\pi = \sum \bigoplus H_j$, where the H_j are mutually isomorphic irreducible representations of $\mathrm{U}(H)_\infty$. Theorem 1 implies that if we set $H_0 = H_\pi^{\mathrm{U}(H\ominus\mathbf{C}^n)}$, then $H_0 = H_\pi^{\mathrm{U}(H\ominus\mathbf{C}^n)_\infty} = \sum H_j \cap H_0$, and for large n these are nonempty spaces by Kirillov's Lemma 3 [**2**]. Since $\mathrm{U}(H) = \mathrm{U}(H\ominus\mathbf{C}^n)\cdot\mathrm{U}(H)_\infty$, we see each H_j is $\mathrm{U}(H)$ invariant. The theorem is clearly true for irreducible representations, and this completes the proof. □

Now suppose we fix a decomposition $H = H_+ \oplus H_-$, where H_+ and H_- are infinite dimensional. Fix $1 \leq p \leq \infty$. The restricted unitary group $\mathrm{U}_{(p)}$ consists of those $g \in \mathrm{U}(H)$ which have the matrix form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with b, c in the Schatten p-class. $\mathrm{U}_{(p)}$ is the unitary group for the following Banach*-algebra: $A = \left\{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{L}(H) : \beta, \gamma \text{ are } p\text{-class}\right\}$, where the norm is

$$\left|\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right|_A = \left|\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}\right|_\infty + \left|\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix}\right|_p .$$

Hence $\mathrm{U}_{(p)}$ is naturally a Banach Lie group. Its identity component consists of those g with index $(a) = 0$.

COROLLARY 1. *Let π be a separable unitary representation of* $\mathrm{U}(H)_\infty \cap \mathrm{U}_{(p)}$. *Then there exists a unique extension of π to a unitary representation of* $\mathrm{U}_{(p)_0}$.

PROOF. $\mathrm{U}_{(p)0}$ is the product of $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$ and $\mathrm{U}_{(p)} \cap \mathrm{U}(H)_\infty$ (if $g \in \mathrm{U}_{(p)_0}$, then because a, d are Fredholm of index zero, they have polar decompositions, implying

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix} \begin{pmatrix} |a| & q_1^{-1} b \\ q_2^{-1} c & |d| \end{pmatrix};$$

also $|a|^2 = 1 - |c|^2 = 1 +$ compact operator).

Now the restriction of π to $\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \in \mathrm{U}(H_+)_\infty \times \mathrm{U}(H_-)_\infty \subset \mathrm{U}_{(p)} \cap \mathrm{U}(H)_\infty$ has a unique extension π_1 to $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$ by Theorem 2. Hence what we must show is that

$$\pi_1(g)\pi(n)\pi_1(g)^{-1} = \pi(gng^{-1}) \tag{1}$$

for $g \in \mathrm{U}(H_+) \times \mathrm{U}(H_-)$ and $n \in \mathrm{U}_{(p)} \cap \mathrm{U}(H)_\infty$. Now we know (1) is true if g is $1 +$ compact. Given an arbitrary g we can find $g_j \in \mathrm{U}(H_+)_\infty \times \mathrm{U}(H_-)_\infty$ such that $g_j \to g$ strongly. Since π_1 is equivalent to a sum of subrepresentations of the mixed tensor algebra of H, we see $\pi_1(g_j) \to \pi_1(g)$ strongly. On the other hand $g_j n g_j^{-1} \to gng$ in the $\mathrm{U}(H)_\infty \cap \mathrm{U}_{(p)}$ topology; this follows easily from Proposition 2.1 of [**7**]. By taking these limits we obtain (1) in general, which proves the corollary. □

It is known that the group $\mathrm{U}(H)_p = \{g \in \mathrm{U}(H) : g = 1 + p\text{-class}\}$ is not of type I [**8**]. Also one can construct continuous families of irreducible unitary representations for $\mathrm{U}_{(p)_0}$, $1 \leq p \leq 2$ [**9**]. However it seems to be unknown whether $\mathrm{U}_{(p)}$ is Type I or otherwise.

3. On Calkin algebra representations. Let H be a separable Hilbert space.

THEOREM 3. *Let π be a *-representaion of $\mathfrak{L}(H)/\mathfrak{L}(H)_\infty$. Then for any normal operator T on H, each point in the spectrum of $\pi(T)$ is an eigenvalue.*

PROOF. We can assume $\pi(1) = 1$. If P is a selfadjoint projection on H with infinite rank and corank, then $v \in \ker \pi(P)$ implies $\pi(\mathfrak{L}(PH))v \equiv 0$ (analogous to Lemma 1). Arguing as in Lemmas 2 and 3 we see that if 0 is in the essential spectrum of T ($\equiv$ spectrum of $\pi(T)$), then 0 is an eigenvalue of $\pi(T)$. □

Here is a second proof of Theorem 3 (this was communicated to me by R. Kadison). If λ is in the essential spectrum of T, choose $\lambda_1 = \lambda, \lambda_2, \ldots$ dense in that spectrum and orthogonal projections $P_1, P_2, \ldots$ with infinite ranks and sum 1. Then

$S = \sum \lambda_i P_i$ has the same essential spectrum as T. Berg's theorem [**10**] implies that $T = gSg^{-1}+$ compact for some $g \in \mathrm{U}(H)$. In any nontrivial representation π of the Calkin algebra, λ will be an eigenvalue for $\pi(S)$, hence for $\pi(T)$.

References

1. R. Kadison, *Infinite unitary groups*, Trans. Amer. Math. Soc. **72** (1952), 386–399.
2. A. Kirillov, *Representations of the infinite dimensional unitary group*, Dokl. Akad. Nauk SSSR **212** (1973), 288–290.
3. G. I. Ol′shanskii, *Unitary representations of the infinite-dimensional classical groups* $\mathrm{U}(p,\infty)$, $\mathrm{SO}_0(p,\infty), \mathrm{Sp}(p,\infty)$ *and the corresponding motion groups*, Funct. Anal. Appl. **12** (1978), 185–195.
4. A. N. Pressley and G. Segal, *Loop groups*, Oxford Univ. Press, 1987.
5. G. Segal, *Faddeev's anomaly in Gauss's law*, preprint.
6. S. Stratila and D. Voiculescu, *Representations of* AF*-algebras and of the group* $\mathrm{U}(\infty)$, Lecture Notes in Math., vol. 486, Springer-Verlag, Berlin and New York, 1975.
7. H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants.* II, Advances in Math. **21** (1976), 1–29.
8. R. Boyer, *Representation theory of the Hilbert-Lie group* $\mathrm{U}(H)_2$, Duke Math. J. **47** (1980), 325–344.
9. D. Pickrell, *Measures on infinite Grassmann manifolds.* II (in preparation).
10. I. Berg, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971), 365–371.
11. I. Segal, *The structure of a class of representations of the unitary group on a Hilbert space*, Proc. Amer. Math. Soc. **8** (1957), 197–203.

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