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TWO QUESTIONS ON HEEGAARD DIAGRAMS OF S^3

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ABSTRACT. We review some of the methods that have been used to recognize S^3 from a Heegaard diagram. We propose a revision of these methods and examine their failure for manifolds different from S^3.

1. A *Heegaard diagram* of a closed, connected 3-manifold will be denoted by $(F; \partial v, \partial w)$, where (M, F) is the *underlying Heegaard splitting*, i.e. F is a closed, connected surface embedded in M, such that the closures of the two components of $M \backslash F$ are handlebodies V and W, and where v and w are complete systems of meridians for V and W. It is assumed also that ∂v cuts ∂w transversally.

A problem important for its relation with the Poincaré conjecture is to decide if a given Heegaard diagram corresponds to S^3 (see, for instance, [**2**]).[1] Due to the fact that the Heegaard splittings of S^3 are canonical [**11**], this problem is reduced to finding if (M, F) has a *trivial handle* by inspecting the diagram $(F; \partial v, \partial w)$. If $(F; \partial v, \partial w)$ has a *cancelling pair*, i.e. curves $\partial v_i, \partial w_j$ which cut each other in a single point, then (M, F) has a trivial handle. But it is easy to show that the converse is not always true.

An important contribution to the study of Heegaard diagrams is due to Singer [**8**] who, among other things, proved that *between two systems of meridian discs v and v' of a handlebody V, there exist a finite sequence of systems*

$$v = v^0, v^1, \ldots, v^n = v'$$

where v^{i+1} comes from v^i by a single Singer move ("geometric T-transformation" in [**10**]), i.e. replacing a disc x of the system v^i by a disc contained in $V \backslash v^i$.

The problem of detecting a trivial handle was approached by Whitehead as follows [**13**]. Let $(F; \partial v, \partial w)$ be a diagram with n cancelling pairs (v_i, w_i), $i = 1, \ldots, n$, such that $\# v \cap (w_1 + \cdots + w_n) = n$, and let $(F; \partial v', \partial w)$ be obtained from $(F; \partial v, \partial w)$ by taking a new system v' in V. Whitehead shows that *it is possible to construct a sequence of systems $v' = v^0, v^1, \ldots, v^m$ such that $\# v^i \cap (w_1 + \cdots + w_n) < \# v^{i-1} \cap (w_1 + \cdots + w_n)$, $i = 1, \ldots, m$; and $w_1, \ldots, w_n$ together with n discs of v^m form n cancelling pairs.* The construction of such a v^i is automatic, once a "cut-point of $(w_1, \ldots, w_n)$ with respect to v^{i-1}" is detected (see [**13**]), and this cut-point always exists, as Whitehead proves.[2]

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[1] That there exists an algorithm to decide this has been announced by W. Haken in his address to the "Workshop on 3-manifolds" 16.I.1985, MSRI. It only remains to find a *practical* one (see "Abstracts from workshop on 3-manifolds" MSRI preprint #07312-85).

[2] A cut-point of the dual diagram was called a "wave" in [**14**].

But not every diagram of (M,F) is of type $(F;\partial v',\partial w)$, because w can also be modified. If this happens, Whitehead's approach fails in general, as Whitehead himself probably knew [**13**, p. 56]. However, for $M = S^3$, where every diagram $(F;v,w)$ comes from one which has (genus of F) cancelling pairs, it was believed [**14**] that either there is a cut-point of w with respect to v, or there is one of v with respect to w. If this were the case, the problem of detecting S^3 would be solved. This, amazingly, is true for the Heegaard diagrams of genus two of S^3 [**4**, **6**], but is false for higher genus (see [**9**, **7**] and two unpublished examples of Ochiai). These examples, however, have cancelling pairs and, therefore, are reducible (though not by Whitehead's procedure). It is natural to ask

Question 1. *Are there Heegaard diagrams of S^3 without "cut-points" and without cancelling-pairs?*

2. Another approach to the problem is due to Haken [**2**], who, using results of Whitehead [**12**] and Zieschang [**15**] (see [**10**]), remarks that given $(F;\partial v,\partial w)$ there exists an algorithm to obtain a v' such that $\#\partial v' \cap \partial w \leq \#\partial v'' \cap \partial w$ for every v''. Once v' is found, the roles of (v',w) are interchanged, and, again using the algorithm, one determines a w', etc., etc., ... until finally one gets a $(F;\partial\hat{v},\partial\hat{w})$ such that

$$\#\partial v' \cap \partial\hat{w} \geq \#\partial\hat{v} \cap \partial\hat{w} \leq \#\partial\hat{v} \cap \partial w'$$

for every $(F;\partial v',\partial w')$. A diagram such as $(F;\partial\hat{v},\partial\hat{w})$ was called *pseudominimal* in [**1**], and we have just said that one such can always be obtained.

Waldhausen [**10**] thought that if $(F;\partial v,\partial w)$ is pseudominimal and if (M,F) has a trivial handle, then $(F;\partial v,\partial w)$ ought to have a cancelling pair. Unfortunately this is false (see [**1** and **5**]). A different, and easier, example is due to Haken [**3**] (see [**16**]). It is the diagram of genus 2 of $L(13,5)$ (Figure 1), that was found by realizing geometrically the group presentation

$$\mathbf{Z}_{13} = |a,b\colon a^3b^{-2} = a^2b^3 = 1|.$$

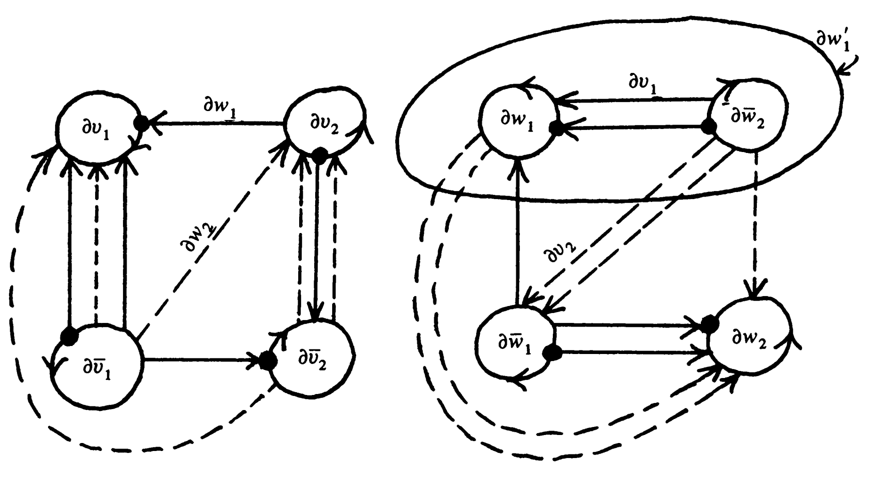

FIGURE 1

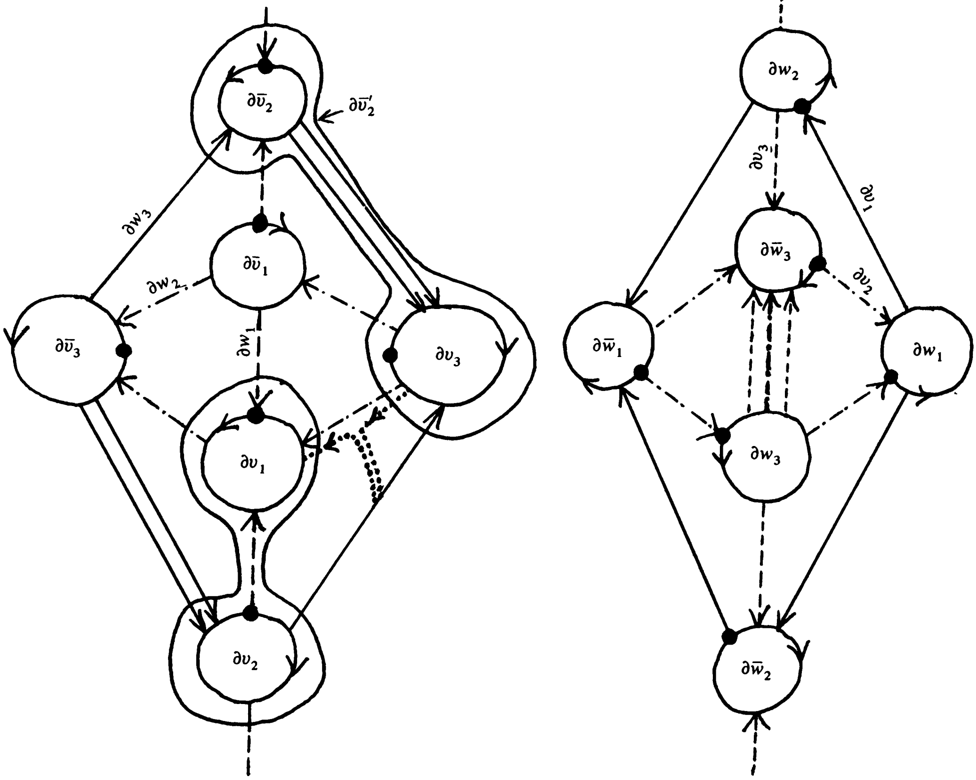

FIGURE 2

The diagram is pseudominimal without cancelling pairs. However the algorithm mentioned at the beginning of this section, applied to $(F; \partial v_1, \partial w)$, where ∂v_1 is a *single* curve, gives w' such that $\#\partial v_1 \cap \partial w' < \#\partial v_1 \cap \partial w$ and such that $\#\partial v_1 \cap \partial w' \leq \#\partial v_1 \cap \partial w''$ for every w''. Using this, we can sharpen the procedure proposed by Haken (*the Haken algorithm*) as follows:

1st step. Get $(F; \partial v, \partial w)$ pseudominimal.

2nd step. Using the algorithm just mentioned, minimize $(F; \partial v, \partial w_i)$ and $(F; \partial v_j, \partial w)$ for every w_i and v_j. If g is the genus of F, the final product of these two steps are $2g$ "diagrams" (one system having g curves, and the other a single curve).

I thought that if (M, F) has a trivial handle, at least one of these $2g$ "diagrams" would exhibit a cancelling pair. And, in fact, this is what happens with the example in [**1**] (see [**5**]) and for the example of Haken (in Figure 1, the curves $(\partial v_1, \partial w_1')$ are a cancelling pair). However the following example shows that this is not true in general:

EXAMPLE. The diagram of Figure 2 is pseudominimal without a cancelling pair, but the underlying Heegaard splitting has genus 2. This can be proved by realizing the two Singer moves (in v and w respectively) sketched at the lower part of Figure 2. The manifold M is the Seifert manifold which is the 2-fold covering of S^3 branched over the torus link $\{3, 9\}$. Realizing the 2nd step of the algorithm we

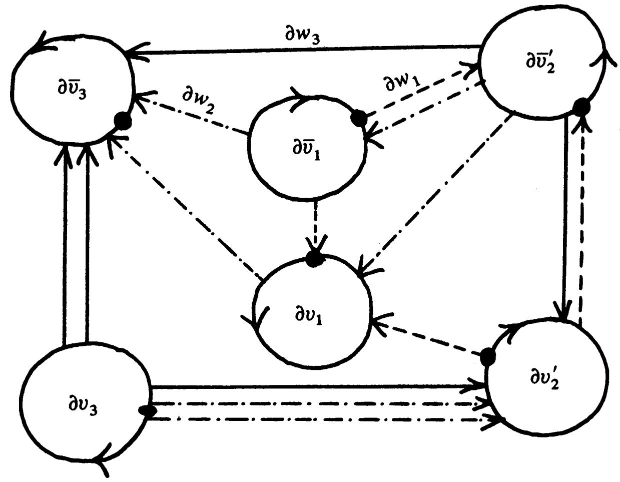

FIGURE 3

obtain six "diagrams," namely:

$$(F; \partial v, \partial w_1), (F; \partial v, \partial w_2), (F; \partial v' = \partial(v_1, v'_2, v_3), \partial w_3)$$
$$\text{(Figure 3)}, (F; \partial v_1, \partial w), (F; \partial v_2, \partial w), (F; \partial v_3, \partial w)$$

and none of them has a cancelling pair.

But still one can ask

Question 2. *Let* $(S^3, F; v, w)$ *be pseudominimal and let* g *be the genus of* F. *Does any of the* $2g$ *"diagrams," obtained from* $(F; \partial v, \partial w_i)$, $(F; \partial v_j, \partial w)$ *by the Haken algorithm, have a cancelling pair?*

REMARK. Lemma 3 (p. 793) of [**12**] implies that it is impossible to reduce $\# \partial v_j \cap \partial w_i$ by a single Singer move applied to any one of the already minimized $2g$ diagrams $(F; \partial v, \partial w_i)$, $(F; \partial v_j, \partial w)$.

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