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UNSTABLE COMPOSITIONS RELATED TO THE IMAGE OF J

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ABSTRACT. We study elements in the homotopy groups of spheres which are desuspensions of generators of the image of the J-homomorphism for $p = 2$. $J_*(\mathbf{R}P^\infty)$ is used to detect the nontriviality of compositions and Toda brackets involving these elements. Some formulas are derived which show how various compositions of these elements are related to one another.

1. The purpose of this paper is to show how the machinery of **[M1]** can be used to detect certain compositions in the unstable homotopy groups of spheres. We begin by stating a result from **[M1]**. All spaces are presumed to be localized at the prime 2.

For any positive integer j, let $n(j)$ denote the smallest integer n for which there is a homotopy class $\rho_j \in \pi_{8j-1}(\Omega^{2n+1}S^{2n+1})$ such that the composite

$$S^{8j-1} \xrightarrow{\rho_j} \Omega^{2n+1}S^{2n+1} \to QS^0$$

represents a generator of the stable image of the J-homomorphism. For any non-negative integer t, let $\nu(t)$ denote the exponent in the highest power of 2 which divides t. We have

THEOREM 1.0. *Let $\nu(j) = 4a + b$, $0 \leq b \leq 3$. Then*

$$n(j) = \begin{cases} 4a+4 & \textit{if } b = 0, \\ 4a+6 & \textit{if } b = 1, \\ 4a+7 & \textit{if } b = 2, \\ 4a+7 & \textit{if } b = 3. \end{cases}$$

The following theorems were first stated in **[M2]**. We need to define a numerical function $m(j,k)$ as follows: write $\nu(j+k) + \nu(j) = 4c + d$, $0 \leq d \leq 3$, and let

$$m(j,k) = \begin{cases} 4c+7 & \text{if } d = 0, \\ 4c+9 & \text{if } d = 1, \\ 4c+10 & \text{if } d = 2, \\ 4c+10 & \text{if } d = 3. \end{cases}$$

Let ρ_j be as above and let k be an integer such that there exists a map

$$\rho_k\colon S^{8(j+k)-2} \to S^{8j-1}$$

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which suspends to a generator of the image of J in dimension $8k-1$. Assume $n(j) \le m \le m(j,k)$.

THEOREM 1.1. *The composite*

$$\rho_j \rho_k \colon S^{8(j+k)-2} \to S^{8j-1} \to \Omega^{2n(j)+1} S^{2n(j)+1} \to \Omega^{2m+1} S^{2m+1}$$

is essential.

Comparing different composites yields the following:

THEOREM 1.2. *Let k, j, and n be such that $\rho_j\rho_k$ and $\rho_k\rho_j$ are both defined in $\pi_{8(j+k)-2}(\Omega^{2n+1}S^{2n+1})$. Then, modulo odd factors and elements with trivial e-invariant, $j\rho_j\rho_k = k\rho_k\rho_j$.*

It is possible to detect higher-order compositions. For example, assume i is such that ρ_i is defined in $\pi_{8(j+k+i)-3}(S^{8(j+k)-2})$ and $\rho_k\rho_i = 0$. Assume m is such that $m > m(j,k)$ and

$$S^{8(j+k)-2} \xrightarrow{\rho_k} S^{8j-1} \xrightarrow{\rho_j} \Omega^{2m+1}S^{2m+1}$$

is null. Then we can form the Toda bracket $\{\rho_j, \rho_k, \rho_i\}$. Now define a numerical function $r(j,k,i)$ by writing $\nu(j+k+i) + \nu(j+k) + \nu(i) = 4e + f$, $0 \le f \le 3$, and set

$$r(j,k,i) = \begin{cases} 4e+12 & \text{if } f = 0, \\ 4e+14 & \text{if } f = 1, \\ 4e+15 & \text{if } f = 2, \\ 4e+15 & \text{if } f = 3. \end{cases}$$

THEOREM 1.3. *If $m \le r \le r(j,k,i)$ then $0 \notin \{\rho_j, \rho_k, \rho_i\}$ in*

$$\pi_{8(j+k+i)-2}(\Omega^{2r+1}S^{2r+1}).$$

Furthermore, if j, k, i and n are such that $\{\rho_j, \rho_k, \rho_i\}$ and $\{\rho_j, \rho_i, \rho_k\}$ are both defined in $\pi_{8(j+k+i)-2}(\Omega^{2n+1}S^{2n+1})$ then, modulo odd factors and elements with trivial e-invariant,

$$(j+k)\{\rho_j, \rho_k, \rho_i\} = (j+i)\{\rho_j, \rho_i, \rho_k\}.$$

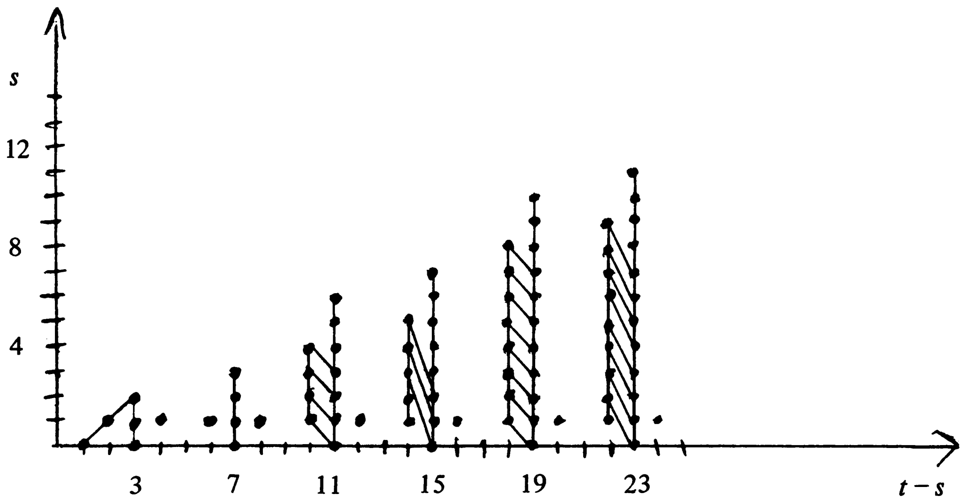
Similar formulas can be easily computed for other permutations and also for nth order brackets.

2. Proof of Theorems 1.1 and 1.2. Recall that the spectrum J is defined as the fiber of a certain map $bo \to \Sigma^4 bsp$ where bo and bsp denote, respectively, real and symplectic connective K-theory. For a detailed treatment of the properties of J, see, for example, [**DM**]. Let P denote the suspension spectrum of $\mathbf{R}P^\infty$. There exists a stable 3-cell complex, denoted by $B(1)$, such that $B(1) \wedge bo = bsp$. In [**M1**], a spectral sequence is constructed which converges to $J_*(P)$ and has as its E_1-term:

$$E_1^{s,t} = \mathrm{Ext}_{A_1}^{s,t}(H^*(P); \mathbf{Z}/2) \oplus \mathrm{Ext}_{A_1}^{s-1,t-4}(H^*(P \wedge B(1)); \mathbf{Z}/2). \tag{2.1}$$

We have $J_{8j-1}(P) = \mathbf{Z}/2^{\nu(j)+4}$ and $J_{8j-2}(P) = \mathbf{Z}/2^{\nu(j)+1}$. These groups are displayed in the above-mentioned spectral sequence as adjacent towers with differentials between them (see diagram (2.2)). The spectral sequence of (2.1) is not obtained from an Adams resolution of $J \wedge P$, but rather is constructed from minimal

Adams resolutions of $bo \wedge P$ and $\Sigma^4 bsp \wedge P$ using a stable version of Proposition 3.3 of [**M1**].



$$\mathrm{Ext}^{s,t}_{A_1}(H^*(P), \mathbf{Z}/2) \oplus \mathrm{Ext}^{s-1,t-4}_{A_1}(H^*(P \wedge B(1)), \mathbf{Z}/2)$$

DIAGRAM (2.2)

Denote by $\Omega^\infty(X)$ the infinite loop space corresponding to a spectrum X. Let $\tilde{\rho}_j \colon S^{8j-1} \to \Omega^\infty(J \wedge P)$ represent a generator of $J_{8j-1}(P)$. Set $n = n(j)$. Then $\tilde{\rho}_j$ factors through $\Omega^\infty(J \wedge P^{2n})$ and Theorem 1.5 of [**M1**] implies that ρ_j can be constructed so that the following diagram commutes.

(2.3)
$$\begin{tikzcd} S^{8j-1} \arrow[r, "\tilde{\rho}_j"] \arrow[d, "\rho_j"] & \Omega^\infty(J \wedge P^{2n}) \\ \Omega^{2n+1} S^{2n+1} \arrow[r, "s"'] & Q(\mathbf{R}P^{2n}) \arrow[u, "h_J"'] \end{tikzcd}$$

where h_J is induced by the map $S^0 \to J$, and s is obtained from the Snaith splitting [S].

Now, the first part of Theorem 1.1 will follow from diagram (2.3) and

PROPOSITION 2.4. *In $J_{8(j+k)-2}(P^{2n})$, where $n = n(j)$, $\tilde{\rho}_j \rho_k$ is nonzero.*

PROOF. Filter P by even-dimensional skeleta so that the cofibers are Moore spectra:

(2.5)
$$\begin{tikzcd} P^2 \arrow[r] \arrow[d] & P^4 \arrow[r] \arrow[d] & P^6 \arrow[r] \arrow[d] & \cdots \arrow[r] & P^{2m} \arrow[r] \arrow[d] & \cdots \arrow[r] & P \\ M^2 & M^4 & M^6 & & M^{2m} & & \end{tikzcd}$$

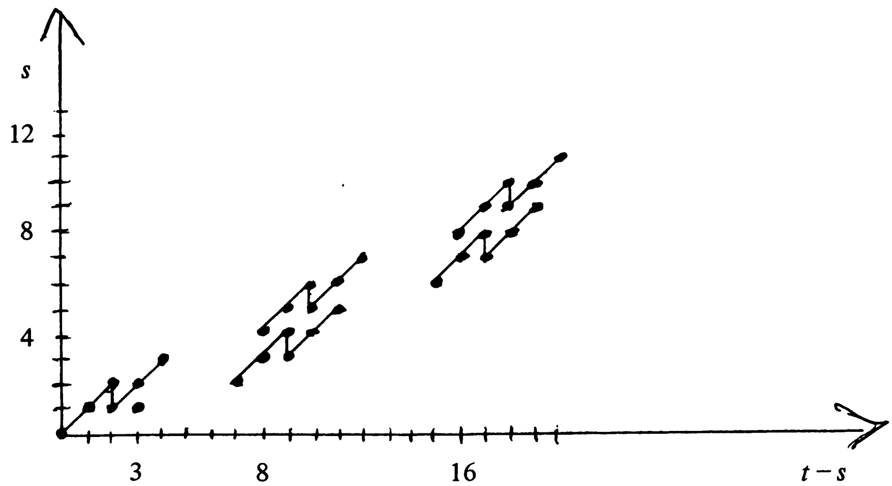
Applying $J_*(\_\_)$ to diagram (2.5) yields an Atiyah-Hirzebruch type spectral sequence converging to $J_*(P)$. Note that $J_*(M^{2m})$ can be obtained from a spectral sequence like (2.1) with P replaced by M^{2m}, and in this case the spectral sequence of (2.1) collapses. Thus we can apply $H^*(\_\_)$ to diagram (2.5) and obtain short exact sequences in cohomology. Then apply

$$\mathrm{Ext}_{A_1}^{s,t}(\_\_;\mathbf{Z}/2)\oplus \mathrm{Ext}_{A_1}^{s-1,t-4}(\_\_\otimes H^*B(1);\mathbf{Z}/2)$$

to obtain a spectral sequence with the first term being:

$$(2.6)\qquad E_0^{s,t,m}=\mathrm{Ext}_{A_1}^{s,t}(H^*(M^{2m}),\mathbf{Z}/2)\oplus \mathrm{Ext}_{A_1}^{s-1,t-4}(H^*(M^{2m}\wedge B(1));\mathbf{Z}/2).$$

The E_1-term of the resulting spectral sequence is a filtered form of (2.1) (see diagram (2.7)).



$$(2.7)\qquad \mathrm{Ext}_{A_1}^{s,t}(H^*(M^1),\mathbf{Z}/2)\oplus \mathrm{Ext}_{A_1}^{s-1,t-4}(H^*(M^1\wedge B(1)),\mathbf{Z}/2)$$

DIAGRAM (2.7)

Now observe that $\tilde{\rho}_j\in J_{8j-1}(P)$ has filtration $s=4j-\nu(j)-4$ where $t-s=8j-1$. Thus we can solve for m and we see that the smallest m such that $\tilde{\rho}_j$ factors through $J_{8j-1}(P^{2m})$ is $n(j)$. We must show that $\tilde{\rho}_j\rho_k$ is essential in $J_{8(j+k)-2}(M^{2n})$, where $n=n(j)$. This will follow from

LEMMA 2.8. *Let α be any element in $J_*(M^1)=bo_*(M^1)\oplus bsp_{*-3}(M^1)$ in the first summand. Then $\alpha\rho_k\in J_*(M^1)$ is nonzero and lies in the second summand.*

PROOF. It is helpful to refer to diagram (2.7). First let $\alpha=\overline{\mu}_s$, where $\overline{\mu}_s$ denotes the element in $J_{8s+1}(M^1)$ which maps nontrivially to $bo_{8s+1}(M^1)$. Note that $\overline{\mu}_s$ is the image of an element $\mu_s\in\pi_{8s+1}S^0$. J. F. Adams proves in [**A**] that in $\pi_*(S^0)$, $\mu_s\rho_k=\rho_{s+k}\eta$. But $\rho_{s+k}\eta$ maps nontrivially into $J_{8(k+s)}(M^1)$ and is the desired element, proving the lemma in this case. To complete the proof of 2.8 observe that the extensions in $bo_*(M^1)$ and $bsp_*(M^1)$ show that if α is any element in $J_*(M^1)=bo_*(M^1)\oplus bsp_{*-3}(M^1)$ in the first summand, $\alpha\rho_k$ is nonzero and is the corresponding element in the second summand.

Now we prove that $\tilde{\rho}_j\rho_k$ is essential in $J_{8(j+k)-2}(P^{2m})$ where $m = m(j,k)$. Observe that the filtration of $\tilde{\rho}_j\rho_k$ in (2.6) is $s = (4j - \nu(j) - 4) + (4k - 2) = 4(j+k) - \nu(j) - 6$. The permanent cycle representing $\tilde{\rho}_j\rho_k$ is hit by a differential whose source has filtration $s = 4(j+k) - 8 - [\nu(j+k) + \nu(j)]$ and stem $t - s = 8(j+k) - 1$. As before, one easily solves for m to find that the source of the differential does not occur until $P^{2(m(j,k)+1)}$.

To prove Theorem 1.2, simply note that when $\tilde{\rho}_j\rho_k$ and $\tilde{\rho}_k\rho_j$ are both defined, they have filtrations $4(j+k) - 6 - \nu(j)$ and $4(j+k) - 6 - \nu(k)$, respectively. Since the extensions are nontrivial in the tower in dimensions $8(j+k) - 2$, this implies that $2^{\nu(j)}\tilde{\rho}_j\rho_k = 2^{\nu(k)}\tilde{\rho}_k\rho_j$ modulo elements of higher filtration, which gives 1.2.

3. Proof of Theorem 1.3. Consider the composite

$$S^{8(j+k+i)-3} \xrightarrow{\rho_i} S^{8(j+k)-2} \xrightarrow{\rho_k} S^{8j-1} \xrightarrow{\tilde{\rho}_j} \Omega^\infty(J \wedge P^{2r}),$$

where $m(j,k) < r \le r(j,k,i)$. By hypothesis, $\rho_k\rho_i = 0$ and by the calculation of §2, $\tilde{\rho}_j\rho_k = 0$. Thus we can form the Toda bracket $\{\tilde{\rho}_j, \rho_k, \rho_i\}$. The idea of the proof is to observe that if α is any element in the bracket $\{\tilde{\rho}_j, \rho_k, \rho_i\}$, then α is represented in the spectral sequence (2.1) by a permanent cycle obtained by composing ρ_i with the source of the differential which hits the element representing $\tilde{\rho}_j\rho_k$.

PROPOSITION 3.1. *$0 \notin \{\tilde{\rho}_j, \rho_k, \rho_i\}$ in $\pi_{8(j+k+i)-2}(\Omega^\infty(J \wedge P^{2r}))$ for $m(j,k) < r \le r(j,k,i)$.*

PROOF. Consider the following commutative diagram:

(3.2)

$$\begin{tikzcd}
 & S^{8(j+k)-2} \arrow[r, "\beta"] \arrow[d, "\rho_k"] & \Omega^\infty(J \wedge \Sigma^{-1}M^{2(m+1)}) \arrow[d] \\
 & S^{8j-1} \arrow[r, "\tilde{\rho}_j"] \arrow[d, "i_1"] & \Omega^\infty(J \wedge P^{2m}) \arrow[d, "i_2"] \\
S^{8(j+k+i)-2} \arrow[r, "\tilde{\rho}_i"] \arrow[dr, "\Sigma\rho_i"'] & S^{8j-1} \cup_{\rho_k} e^{8(j+k)-1} \arrow[r, "\phi"] \arrow[d, "P_1"] & \Omega^\infty(J \wedge P^{2(m+1)}) \arrow[d, "P_2"] \\
 & S^{8(j+k)-1} \arrow[r, "\Sigma\beta"] & \Omega^\infty(J \wedge M^{2(m+1)})
\end{tikzcd}$$

Here, $m = m(j,k)$ and the first vertical column is a cofibration; the second is a fibration. An extension ϕ exists since $i_2\tilde{\rho}_j\rho_k = 0$ and β represents the source of the differential which hits $\tilde{\rho}_j\rho_k$. A coextension $\tilde{\rho}_i$ exists since $\rho_k\rho_i = 0$. The composite $\phi\tilde{\rho}_i$ is an element of $\{\tilde{\rho}_j, \rho_k, \rho_i\}$. We wish to show $P_2\phi\tilde{\rho}_i \neq 0$. But Lemma 2.8 implies that $\Sigma\beta\Sigma\rho_i \neq 0$, and so this proves $0 \notin \{\tilde{\rho}_j, \rho_k, \rho_i\}$ in

$$\pi_{8(j+k+i)-2}(\Omega^\infty(J \wedge P^{2(m+1)})).$$

Now observe that the filtration of $\beta\rho_i$ is $s = 4(j+k+i) - [\nu(j+k) + \nu(j)] - 10$. Hence, the source of the differential hitting $\beta\rho_i$ has filtration $s = 4j + 4k + 4i - [\nu(j+k+i) + \nu(j+k) + \nu(j)] - 12$, and $t - s = 8(j+k+i) - 2$. Again solving for the dimension of the skeleton of P, we find that this differential does not occur until we include into $\Omega^\infty(J \wedge P^{2(r+1)})$, where $r = r(j,k,i)$. Finally, by comparing the filtrations of $\{\tilde{\rho}_j, \rho_k, \rho_i\}$ and $\{\tilde{\rho}_j, \rho_i, \rho_k\}$ we get the second statement of 1.3.

REFERENCES

[**A**] J. F. Adams, *On the groups* $J(X)$*-IV*, Topology **5** (1966), 21–71.

[**DM**] D. M. Davis and M. Mahowald, v_1*- and* v_2*-periodicity in stable homotopy theory,* Amer. J. Math. **103** (1981), 615–619.

[**M1**] Mark Mahowald, *The image of J in the EHP sequence*, Ann. of Math. (2) **116** (1982), 65–112.

[**M2**] ——, *Descriptive homotopy of the elements in the image of the J-homomorphism*, Proc. Tokyo Conf. Manifolds, 1973, pp. 255–263.

[**S**] V. P. Snaith, *A stable decomposition of* $\Omega^n S^n X$, J. London Math. Soc. (2) **7** (1974), 577–583.

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