PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

SHORTER NOTES

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VECTOR MEASURES WITH VALUES IN THE COMPACT OPERATORS

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(Communicated by William J. Davis)

ABSTRACT. As a consequence of a recent result due to Kaftal and Wiess it is shown that any vector measure (for the strong operator topology) with values in the space of compact operators on a Hilbert space is σ-additive for the uniform operator topology. This leads to an elegant and simple proof of a result due to Diestel and Faires on the uniform operator σ-additivity of the indefinite integral induced by a compact selfadjoint operator.

Let H be a Hilbert space, $B(H)$ the space of all continuous linear operators from H into itself and $K(H)$ the ideal of all compact operators. The following result was recently announced in [**3**]; the notation and definitions are from there. In particular, $\mathbf{N}$ is the set of all natural numbers and, given operators K_n, $n \in \mathbf{N}$, the series $\sum_{n=1}^{\infty} K_n$ denotes the limit in the strong operator topology (briefly, S.O.T.) of the sequence of (ordered) partial sums $\sum_{n=1}^{M} K_n$, $M \in \mathbf{N}$.

THEOREM. *Let $K_n \in B(H)$, $n = 1, 2, \ldots$, be such that $\sum_{n\in\mathbf{N}} K_n$ converges unconditionally with respect to the S.O.T. and $\sum_{n\in F} K_n \in K(H)$ for every subset $F \subseteq \mathbf{N}$. Then $\sum_{n=1}^{\infty} K_n$ converges in the uniform operator topology* (*briefly, U.O.T.*).

REMARK 1. In [**3**] the authors ask whether this result remains valid if the S.O.T. is replaced by the weak operator topology? The answer is yes. Indeed, if $B_s(H)$ denotes the linear space $B(H)$ equipped with the S.O.T., then $B_s(H)$ is a quasicomplete locally convex Hausdorff space whose weak topology, as a locally convex space, is precisely the weak operator topology. Accordingly, the Orlicz-Pettis lemma applies: it states that a series in a locally convex space is unconditionally convergent if and only if it is unconditionally convergent in the weak topology.

REMARK 2. The conclusion of the theorem can be strengthened to state that the series $\sum_{n=1}^{\infty} K_n$ actually converges unconditionally with respect to the U.O.T. In fact, noticing that whenever the hypotheses of the theorem hold for a given sequence

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Received by the editors March 9, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28B05, 46G10; Secondary 47B05.

K_n, $n = 1, 2, \ldots$, then they also hold for any rearrangement $K_{\pi(n)}$, $n = 1, 2, \ldots$, of the given sequence (π is a permutation of $\mathbf{N}$), we can conclude that $\sum_{n=1}^{\infty} K_{\pi(n)}$ converges in the U.O.T. Since π is arbitrary the conclusion follows.

PROPOSITION 1. *Let $m\colon \Lambda \to B_s(H)$ be a vector measure with domain a σ-algebra Λ. If the range $m(\Lambda) = \{m(E); E \in \Lambda\}$ of m is contained in $K(H)$, then m is σ-additive with respect to the U.O.T.*

PROOF. Let $\{E_n\}$ be a sequence of disjoint sets in Λ. Then the sequence $K_n = m(E_n)$, $n = 1, 2, \ldots$, satisfies the hypotheses of the Theorem and hence, Remark 2 applies to the series $\sum_{n \in \mathbf{N}} m(E_n)$. □

For the case of compact selfadjoint operators the following result is due to Diestel and Faires [**1**, Corollary 1.4]. Their proof was simplified by Shuchat [**4**, Proposition 9] and extended to include compact, scalar-type spectral operators; all the definitions and facts used about scalar-type spectral operators can be found in [**2**]. A particularly simple and natural proof can be based on Proposition 1.

PROPOSITION 2. *Let $T \in B(H)$ be a compact, scalar-type spectral operator, in which case $T = \int_{\sigma(T)} \lambda \, dP(\lambda)$, where $P\colon \mathcal{B}(\sigma(T)) \to B_s(H)$ is the resolution of the identity for T and $\mathcal{B}(\sigma(T))$ is the σ-algebra of Borel subsets of the spectrum, $\sigma(T)$, of T. Then the indefinite integral induced by T, namely, the set function*

$$Q_T\colon E \to \int_E \lambda \, dP(\lambda), \qquad E \in \mathcal{B}(\sigma(T)),$$

is σ-additive with respect to the U.O.T.

PROOF. Since P is σ-additive for the S.O.T. and the identity function λ on $\sigma(T)$ is P-integrable, the Orlicz-Pettis lemma implies that Q_T is also σ-additive for the S.O.T. That Q_T assumes its values in $K(H)$ is clear from the identity $Q_T = P(E)T = TP(E)$, valid for every $E \in \mathcal{B}(\sigma(T))$. □

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