CHARACTERIZATIONS OF DENTING POINTS
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ABSTRACT. Let \( x \) be a PC (point of continuity) for a bounded closed convex set \( K \) of a Banach space. Then \( x \) is a denting point of \( K \) if and only if \( x \) is an extreme point (resp. strong extreme point; weak*-extreme point) of \( K \). A new definition for denting point is also given.

Throughout this paper, \( K \) is a bounded closed convex set of a Banach space \( X \). An element \( x \) in \( K \) is called a denting point of \( K \) if \( x \in \partial(K \setminus B(x, \varepsilon)) \) for all \( \varepsilon > 0 \) where \( B(x, \varepsilon) = \{ y : y \in X, \| y - x \| < \varepsilon \} \). \( x \) is a PC (point of continuity) for \( K \) if the identity mapping \((K, \text{weak}) \rightarrow (K, \text{norm})\) is continuous at \( x \).

Recall that \( x \) is a strong extreme point of \( K \) if for any sequences \( \{y_n\} \) and \( \{z_n\} \) in \( K \), \( \lim_{n \to \infty} \frac{1}{2}(y_n + z_n) - x \| = 0 \) implies that \( \lim_{n \to \infty} \| y_n - x \| = 0 \). \( x \) is a weak*-extreme point of \( K \) if \( x \) is an extreme point of \( K \) where \( K \) is the weak*-closure of \( K \) in \( X^{**} \). A Banach space \( X \) is said to have the Kadec property (K) (resp. property (G); midpoint locally uniform rotundity) if every point on the unit sphere of \( X \) is a PC (resp. denting point; strong extreme point) for the closed unit ball of \( X \) [LL, T]. It has been proved [LLT1] that a Banach space \( X \) has property (G) if and only if \( X \) is strictly convex and \( X \) has property (K). It is easy to see that every denting point of \( K \) is a strong extreme point of \( K \) and it is known [KR] that every strong extreme point of \( K \) is a weak*-extreme point of \( K \).

**DEFINITION.** An element \( x \) in \( K \) is called a very strong extreme point of \( K \) if for every sequence \( \{x_n\} \) of \( L^1 \)-valued Bochner integrable functions on \([0, 1]\), the condition

\[
\lim_{n \to \infty} \left\| \int_0^1 x_n(t) dt - x \right\| = 0 \quad \text{implies} \quad \lim_{n \to \infty} \int_0^1 \| x_n(t) - x \| dt = 0.
\]

It is well known that if \( x \) is a denting point of \( K \) then \( x \) is a PC for \( K \) and \( x \) is an extreme point of \( K \). In [LLT2] (or see [R]), we show that the converse is true. In fact, we have the following results.

**THEOREM.** Let \( x \) be an element in a bounded closed convex set \( K \) of a Banach space. Then the following are equivalent:

(i) \( x \) is a denting point of \( K \).

(ii) \( x \) is a very strong extreme point of \( K \).

(iii) \( x \) is a PC for \( K \) and \( x \) is an extreme point (resp. strong extreme point, weak*-extreme point) of \( K \).

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A subset \( S \) of \( K \) is an open slice if there are a nonzero \( x^* \) in \( X^* \) and \( \varepsilon > 0 \) such that \( S = \{ x : x \in K, x^*(x) > \sup x^*(K) - \varepsilon \} \). It is well known that \( x \) is a denting point of \( K \) if and only if for any \( \varepsilon > 0 \) there exists a slice \( S \) of \( K \) such that \( x \in S \) and diameter of \( S \) is less than \( \varepsilon \). From the following lemma, if \( x \) is a PC for \( K \) and if \( x \) is a weak*-extreme point of \( K \) then \( x \) is a denting point of \( K \). The authors wish to thank Professor H. P. Rosenthal for providing the lemma. We include a proof of the lemma for the sake of completeness.

**Lemmas.** Let \( x \) be an element in a bounded closed convex set \( K \) of a Banach space. Then \( x \) is a weak*-extreme point of \( K \) if and only if the open slices containing \( x \) form a neighborhood base for \( x \) in the weak topology on \( K \).

**Proof.** Suppose \( x \) is a weak*-extreme point of \( K \). Let \( V = \{ y : y \in K, x^*(y) - x^*(x) < \varepsilon, i = 1, 2, \ldots, n \} \) be a weak neighborhood of \( x \) in \( K \). For \( i = 1, 2 \), let \( M_i = \{ y : y \in K, x^*(y) - x^*(x) > \varepsilon \} \). Suppose \( x \in \overline{c(M_1 \cup M_2)} \). Then there exist \( y_n \in M_1, z_n \in M_2 \) and \( \alpha_n \in [0, 1] \) such that \( \lim_{n \to \infty} \| \alpha_n y_n + (1 - \alpha_n) z_n - x \| = 0 \). Since \( \tilde{K} \) is weak*-compact, we conclude that there exist \( y, x \in \tilde{K} \) and \( 0 < \alpha < 1 \) such that \( x = \alpha y + (1 - \alpha) x \). This contradicts the fact that \( x \) is a weak*-extreme point of \( K \). Thus \( x \notin \overline{c(M_1 \cup M_2)} \). By the Hahn-Banach theorem, there exist \( x^* \in X^* \) and \( \delta > 0 \) such that \( x^*(x) + \delta > \sup x^*(\overline{c(M_1 \cup M_2)}) \). Hence if \( y \) is an element in the open slice \( S = \{ y : y \in K, x^*(y) > \sup x^*(K) - \delta \} \) then \( x^*(y) - x^*(x) < \varepsilon, i = 1, 2 \). Repeating the argument, we conclude that there is an open slice which contains \( x \) and is contained in \( V \). The inverse implication is obvious. □

**Proof of Theorem.** (i)⇒(ii). Suppose \( x \) is a denting point of \( K \). Let \( \{ x_n \} \) be a sequence of \( K \)-valued Bochner integrable functions on \([0, 1] \) such that \( \lim_{n \to \infty} \| \int_0^1 x_n(t) dt - x \| = 0 \). Since for every \( \varepsilon > 0 \) there exists a simple function \( y_n(t) \) such that \( \| x_n(t) - y_n(t) \| < \varepsilon \) for all \( t \), we may assume that \( x_n(t) \) is a simple function. It follows that \( \int_0^1 x_n(t) dt = \sum_{i=1}^{k_n} a_i(n) x_i,n \) where \( x_i,n \in K, a_i(n) > 0 \) and \( \sum_{i=1}^{k_n} a_i(n) = 1, i = 1, 2, \ldots, k_n, n = 1, 2, \ldots \). Suppose that

\[
\lim_{n \to \infty} \int_0^1 \| x_n(t) - x \| dt = \lim_{n \to \infty} \sum_{i=1}^{k_n} a_i(n) \| x_i,n - x \| \neq 0.
\]

By choosing a subsequence if necessary, there exists \( \varepsilon > 0 \) such that \( \sum_{i \in I_n} a_i(n) > \varepsilon \) where \( I_n = \{ i : 1 \leq i \leq k_n, \| x_i,n - x \| \geq \varepsilon \} \). Let \( \lambda_n = \sum_{i \in I_n} a_i(n), \ y_n = \lambda_n^{-1} \sum_{i \in I_n} a_i(n) x_i,n \) and \( z_n = (1 - \lambda_n)^{-1} \sum_{i \notin I_n} a_i(n) x_i,n \). Then \( y_n, z_n \in K \) and \( \sum_{i=1}^{k_n} a_i(n) x_i,n = \lambda_n y_n + (1 - \lambda_n) z_n \). Since \( x \) is a denting point of \( K \), \( \| y_n - x \| > \delta, n = 1, 2, \ldots \) for some \( \delta > 0 \). This contradicts the fact that every denting point is a strong extreme point of \( K \).

It is clear that if \( x \) is a very strong extreme point of \( K \) then \( x \) is a strong extreme point of \( K \). On the other hand, it is proved in [LLT2] that if \( x \) is a PC for \( K \) and \( x \) is an extreme point of \( K \) then \( x \) is a strong extreme point of \( K \). It is known that \( x \) is a weak*-extreme point if \( x \) is a strong extreme point [KR]. Finally, we use the lemma to conclude that \( x \) is a denting point of \( K \). □

**Remark 1.** A Banach space \( X \) is said to have the CPCP (convex point of continuity property) if every bounded closed convex set \( K \) contains at least one PC. It is known [S] that if \( X \) has CPCP then KMP is equivalent to RNP. (Recall...
a Banach space has KMP (resp. RNP) if every bounded closed convex subset of $X$ contains at least one extreme (resp. denting) point.)

**Remark 2.** It can be proved directly that if $x$ is a PC for $K$ and $x$ is a strong extreme point of $K$ then $x$ is a very strong extreme point of $K$.

**Remark 3.** Let $F$ be a total subspace in $X^*$, and let $\tau$ be the locally convex topology on $X$ defined by $F$. By replacing $X^*$ by $F$ (resp. the weak topology by $\tau$), we may define the $\tau$-denting point (resp. $\tau$-PC point) of a bounded closed convex set $K$ in $X$. It can be proved that every $\tau$-PC, extreme point of $K$ is a strong extreme point of $K$ and that every $\tau$-PC, strong extreme point of $K$ is a $\tau$-denting point of $K$.

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