Let $E$ be a locally convex Hausdorff topological vector space, $S$ a nonempty subset of $E$ and $p$ a continuous seminorm on $E$. It is a well-known result (see the proof in Sehgal [8] or Ky Fan [1]) that if $S$ is compact and convex and $f: S \rightarrow E$ is a continuous map, then there exists an $x \in S$ satisfying

$$p(fx - x) = d_p(fx, S) = \min \{ p(fx, y) \mid y \in S \}. \tag{1}$$

Since then a number of authors have provided either an extension of the above theorem to set valued mappings or have weakened the compactness condition therein. Some of these results are

(a) REICH (1978). If $S$ is approximatively compact and $f: S \rightarrow E$ is continuous with $f(S)$ relatively compact, then (1) holds [5].

(b) LIN (1979). If $S$ is a closed unit ball of a Banach space $X$ and $f: S \rightarrow X$ is a continuous condensing map, then (1) holds when $p$ is the norm on $X$ [4].

(c) WATERS (1984). If $S$ is a closed and convex subset of a uniformly convex Banach space $E$ and $f: S \rightarrow 2^E$ is a continuous multifunction with convex and compact values and $f(S)$ is relatively compact, then (1) holds [9].

(d) SEHGAL AND SINGH (1985). Let $S \subseteq E$ with $\text{int}(S) \neq \emptyset$ and $\text{cl}(S)$ convex and let $f: S \rightarrow 2^E$ be a continuous condensing multifunction with convex, compact values and with a bounded range. Then for each $w \in \text{int}(S)$, there exists a continuous seminorm $p = p(w)$ satisfying (1) [6].

Our aim in this presentation is to prove (a) for multifunctions and derive some results as easy corollaries.

For definitions and terminologies we refer to Reich [5] (see also [3]).

**Definition.** A subset $S$ of $E$ is approximatively $p$-compact iff for each $y \in E$ and a net $\{x_\alpha\}$ in $S$ satisfying $p(x_\alpha - y) \rightarrow d_p(y, S)$ there is a subnet $\{x_\beta\}$ and an $x \in S$ such that $x_\beta \rightarrow x$.

Clearly a compact set in $E$ is approximatively norm compact. The converse, however, may fail. For example, the closed unit ball of an infinite dimensional uniformly convex Banach space is approximatively norm compact but not compact.

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Some consequences of the definition follow.

1. An approximatively $p$-compact set $S$ in $E$ is closed. Let $y$ be a cluster point of $S$ and let a net $\{x_\alpha\} \subseteq S$ satisfy $p(x_\alpha - y) \to d_p(y, S) = 0$. Since $S$ is approximatively $p$-compact, $\{x_\alpha\}$ contains a subnet $x_\beta \to x \in S$. Since $x_\beta \to y$ also and $E$ is Hausdorff, $x = y \in S$.

2. If $S$ is a closed and convex subset of a uniformly convex Banach space then $S$ is approximatively norm compact.

Let $y \in E$ and, without loss of generality, assume a sequence $\{x_n\} \subseteq S$ satisfies $\|x_n - y\| \to d(y, S) \equiv \inf\{\|y - x\| \mid x \in S\}$. This implies that $\sup\|x_n\| < \infty$. Consequently, since $S$ is closed and convex, there exist an $x \in S$ and a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} \to x$ weakly. Thus

\[ x_{n_k} - y \to x - y \text{ weakly.} \]

It follows from (*) that

\[ \|x - y\| \leq \lim \|x_{n_k} - y\| = d(y, S), \]

i.e. $d(y, S) = \|x - y\|$.

Consequently, by the definition of the sequence $\{x_n\}$

\[ \|x_{n_k} - y\| \to \|x - y\|. \]

Since $E$ is uniformly convex, (*) and (**) imply that $x_{n_k} - y \to x - y$. This yields $x_{n_k} \to x \in S$. Thus $S$ is approximatively norm compact.

**DEFINITION.** Let $E$ and $F$ be topological vector spaces and let $2^F$ denote the family of nonempty subsets of $F$. The mapping $T: E \to 2^F$ is upper semicontinuous (u.s.c.) iff $T^{-1}(B) = \{x \in E \mid Tx \cap B \neq \emptyset\}$ is closed for each closed subset $B$ of $F$.

3. If $S$ is an approximatively $p$-compact subset of $E$ then for each $y \in E$, $Q(y) = \{x \in S \mid p(y - x) = d_p(y, S)\}$ is nonempty and the mapping defined by $y \to Q(y)$ is an upper semicontinuous (u.s.c.) multifunction on $E$. For a proof see Reich [5].

Note that if $E$ is a uniformly convex Banach space the above projection map $Q$ is single valued and continuous.

Now we give our main result.

**THEOREM 1.** Let $S$ be an approximatively $p$-compact, convex subset of $E$ and let $F: S \to 2^E$ be a continuous multifunction with closed and convex values. If $FS = \bigcup\{Fx \mid x \in S\}$ is relatively compact then there exists an $x \in S$ with

\[ d_p(x, Fx) = d_p(Fx, S). \]

Further, if $d_p(x, Fx) > 0$, then $x \in \partial S$.

Note that $d_p(A, B) = \inf\{p(x - y) \mid x \in A, y \in B\}$. The proof of the above theorem uses the following lemma, whose proof is given in Sehgal and Singh [7, Lemma 2, p. 92].

**LEMMA.** Under the hypotheses of Theorem 1, the mapping $g: S \to R$ (reals) defined by $g(x) = d_p(Fx, S)$ is continuous.

**PROOF OF THEOREM 1.** Define a mapping $G: S \to 2^S$ by

\[ G(x) = \bigcup\{Q(y) \mid y \in Fx, d_p(Fx, S) = d_p(y, S)\}. \]

Note that since $Fx$ is compact, $G(x) \neq \emptyset$. 

...
Further, since $F_x$ is convex, it follows that $G_x$ is also convex. In fact, if $u$ and $v$ are in $G_x$, then there exist elements $y_1$ and $y_2$ in $F_x$ such that $u$ is in $F_{y_1}$ and $v$ is in $F_{y_2}$ and

$$p(y_1 - u) = d_p(y, S) = d_p(F_x, S) = d_p(y_2, S) = p(y_2 - v).$$

Let $t \in [0, 1]$, $w(t) = tu + (1 - t)v$ and $y_3 = ty_1 + (1 - t)y_2$. Then $w(t) \in S$, $y_3$ is in $F_x$ and

$$d_p(y_3, S) \leq p(y_3 - w(t)) \leq tp(y_1 - u) + (1 - t)p(y_2 - v) = d_p(F_x, S) \leq d_p(y_3, S).$$

This implies that

$$d_p(y_3, S) = p(y_3 - w(t)) = d_p(F_x, S).$$

Consequently it follows that for any $t \in [0, 1]$,

$$w(t) \in Q(y_3) \cap G_x;$$

that is, $G_x$ is convex.

Also, since for each $x \in S$,

$$G_x = Q F_x \cap \{y \in F_x | d_p(F_x, S) = d_p(y, S)\},$$

and $Q$ is an u.s.c. function, it follows that $G_x$ is a closed (in fact, compact) subset of $S$.

We show that $G$ is an u.s.c. multifunction. To prove this, we show that $G^{-1}(A)$ is closed for any closed subset $A$ of $S$. Let $\{x_\alpha\} \subseteq G^{-1}(A)$ be a net such that $x_\alpha \to x_0 \in S$. Since $G(x_\alpha) \cap A \neq \emptyset$, choose for each $\alpha$, $z_\alpha \in G x_\alpha \cap A$. It then follows from the definition of $G$ that for each $\alpha$, there is a $y_\alpha \in F x_\alpha$, with $d_p(F x_\alpha, S) = d_p(y_\alpha, S)$ and $z_\alpha \in Q(y_\alpha)$. Since $S$ is compact and $\{y_\alpha\} \subseteq F_S$, without loss of generality we may assume that $y_\alpha \to y_0 \in E$. Further, $F$ being u.s.c., it follows that $y_0 \in F x_0$. Also, since $Q$ is u.s.c., $Q(cl(F_S))$ is compact and since for each $\alpha$, $x_\alpha \in Q(y_\alpha) \subseteq Q(F x_\alpha) \subseteq Q(cl(F_S))$, we may again assume $z_\alpha \to z_0 \in Q(y_0)$. Now, $d_p(y_\alpha, S) \to d_p(y_0, S)$ and by the lemma $d_p(F x_\alpha, S) \to d_p(F x_0, S)$. This implies that $d_p(y_0, S) = d_p(F x_0, S)$ and that $z_0 \in G(x_0) \cap A$, i.e., $x_0 \in G^{-1}(A)$. Thus $G$ is u.s.c. It now follows by a theorem of Himmelberg [2] that there is an $x \in S$ with $x \in G(x)$. This implies that $x \in Q(y)$ for some $y \in F x$ with $d_p(F x, S) = d_p(y, S)$. Now, since $d_p(x, F x) \leq p(x - y) = d_p(y, S) = d_p(F x, S) \leq d_p(x, F x)$, we have $d_p(x, F x) = d_p(F x, S)$.

If $d_p(x, F x) > 0$ then $F x \cap S = \emptyset$. Choose a point $y \in F x$ such that $d_p(x, F x) = p(x - y)$. If $x$ is an interior point of $S$, then the convexity of $S$ implies the existence of a $z \in \partial S$ such that $p(z - y) < d_p(x, F x)$. This implies that $d_p(F x, S) \leq p(z - y) < d_p(x, F x)$, which gives a contradiction. Consequently in this case $x \in \partial S$.

Note that in view of consequence (2), the result due to Waters is a special case of Theorem 1.

The following simple example is due to Waters [9] and shows that even in the special case of the uniformly convex Banach space $E$, continuity therein cannot be replaced by u.s.c. alone.

**EXAMPLE.** Let $E = R^2$ with the Euclidean norm and let $S = [0, 1] \times \{0\}$. Clearly $S$ is convex and compact.
Define $F : S \to 2^E$ by

$$F(a, 0) = \begin{cases} (0, 1) & \text{if } a \neq 0, \\ L = \text{the line segment } [(0, 1), (1, 0)] & \text{if } a = 0. \end{cases}$$

Then for any $A \subseteq E$,

$$F^{-1}(A) = \begin{cases} \emptyset & \text{if } A \cap L \neq \emptyset, \\ S & \text{if } (0, 1) \in A, \\ (0, 0) & \text{if } (0, 1) \notin A, \ A \cap L = \emptyset. \end{cases}$$

Thus $F$ is an u.s.c. but not a l.s.c. multifunction and $FS$ is compact.

However, for any $(a, 0)$,

$$d((a, 0), F(a, 0)) > 1 = d(F(a, 0), S) \quad \text{if } a \neq 0,$$

$$= \frac{\sqrt{2}}{2} \neq d(F(0, 0), S) = 0 \quad \text{if } a = 0.$$ 

Thus $F$ does not satisfy the conclusion of Theorem 1.

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REFERENCES


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