REMOVABLE SINGULARITIES IN THE NEVANLINNA CLASS
AND IN THE HARDY CLASSES
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ABSTRACT. We show that certain sets in C^n, n ≥ 2, which we call n-small,
are removable singularities for holomorphic functions in the Nevanlinna class.
Since our class of sets includes polar sets (in R^{2n}) our result includes the
previous removable singularity results for the Nevanlinna class. We give also
a related result for a subclass of the Hardy class.

1. Throughout this paper, G is an open set in C^n, n ≥ 1, and E is closed
in G. Parreau [12, Théorème 20, p. 182] gave essentially the following result: If
n = 1, E is polar, and f is a holomorphic function in G\E such that log^+ |f|
has a harmonic majorant in G\E, then f has a meromorphic extension to G. In
answering a question of Cima and Graham [2, Remarks 7.4, p. 255], Parreau’s
theorem was in [7, Theorem 3.4, p. 477] extended to the case n ≥ 2 and E polar
in R^{2n}.

The purpose of this note is to give a result, Theorem 1 below, which contains
the above result as a special case. In Theorem 1 the exceptional set E is allowed
to be slightly larger, that is n-small (for the definition see [14, Definition 2.2, p.
101] or §3 below). However, we must then replace the condition that log^+ |f|
has a harmonic majorant in G\E by the condition that the (Riesz) measure Δ log^+ |f|
has locally finite mass near the exceptional set E. In addition, we give in §6 a slight
generalization to [8, Corollary 3.6, p. 301].

2. Let u be a subharmonic function in G\E. One says that the (Riesz) measure
Δu has locally finite mass near E, if Δu(D\E) < ∞ for each open set D ⊂ G
(relatively compact in G). From [1, p. 283] (see also [7, Lemma 3.3, p. 476]), it
follows that if Δu has locally finite mass near E, then u has locally a harmonic
majorant near E (that is, for each open set D ⊂ G there is a harmonic function h
in D\E such that u ≤ h in D\E). The converse holds in the important case when
E is polar in R^{2n} (see [1, p. 283] or the proof of Corollary 1 below), but not in
general. To get a simple example, set E' = {z = x + iy ∈ C||x| = |y|}, u(z) =
− log(max{|x|, |y|}) and h(z) = log √2 − log |z|. Then in C\E' the subharmonic
function u has the harmonic majorant h. However, one sees easily that Δu does
not have locally finite mass near E'.

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3. For $F \subset \mathbb{C}$, set $C^1(F) = \text{cap}^* F$, where $\text{cap}^*$ is the outer logarithmic capacity (for the definition of $\text{cap}^*$, see [5, pp. 210, 273]). If $F \subset \mathbb{C}^n$, $n \geq 2$, then set

$$C^n(F) = \max_{1 \leq j \leq n} H_2\{z_j \in \mathbb{C}|C_n^{-1}\{z_j, Z_j\} \subset F\} > 0.$$ 

Here, and in the sequel, $z = (z_1, \ldots, z_j, \ldots, z_n) = (z_j, Z_j)$, $1 \leq j \leq n$, and $H_\alpha$ is the $\alpha$-dimensional Hausdorff outer measure. We say that $F \subset \mathbb{C}^n$ is $n$-small, if $C^n(F) = 0$.

**Lemma 1** [7, Proposition 2.3, p. 472]. An $\mathcal{F}_\alpha$-set $F \subset \mathbb{C}^n$, $n \geq 2$, is $n$-small if and only if for each $j$, $1 \leq j \leq n$, $H_{2n-2}(F_j) = 0$, where

$$F_j = \{z_j \in \mathbb{C}^{n-1}|\text{cap}^*\{z_j \in \mathbb{C}|(z_j, Z_j) \in F\} > 0\}.$$

With the aid of this lemma it follows from a result of Mattila [10, Corollary 3.3, p. 263] (see also [15, Lemma 6, p. 115]) that polar sets in $\mathbb{R}^{2n}$ are $n$-small.

If the Hausdorff measure $H_2$ in (A) is replaced by the outer logarithmic capacity $\text{cap}^*$, a set function which is sometimes denoted by $g_n$ is obtained. Sets $E$ for which $g_n(E) = 0$ have been used as exceptional sets at least in [1, 16, 17, and 14]. If $F \subset \mathbb{C}^n$ and $g_n(F) = 0$, then clearly $F$ is $n$-small. On the other hand, there are sets $F \subset \mathbb{C}^n$ with $g_n(F) = 0$ which are not even polar. In fact, with the help of [3, Theorem 2, p. 118] one can construct a compact set $F \subset \mathbb{C}^2$ of Hausdorff dimension 4 such that $g_2(F) = 0$ and such that for each $z \in \mathbb{C}$ the sections $F \cap (\mathbb{C} \times \{z\})$ and $F \cap (\{z\} \times \mathbb{C})$ contain at most one point.

4. Next we give the lemmas we need in the sequel.

**Lemma 2** [18, Theorem 1.2, p. 16]. Let $G'$ be a domain of $\mathbb{C}^{n-1}$, $n \geq 2$, and $F \subset G'$. Suppose $j \in \mathbb{N}$, $1 \leq j \leq n$, $r'_j, r''_j \in \mathbb{R}$, $0 < r'_j < r''_j$, and $z_j^0 \in \mathbb{C}$. Let $f$ be a holomorphic function in the open set

$$V(G', z_j^0, r'_j, r''_j) = \{z = (z_j, Z_j) \in \mathbb{C}^n|z_j \in A(z_j^0, r'_j, r''_j), Z_j \in G'\}$$

such that for each $Z_j \in F$ the holomorphic function $f_{Z_j}: A(z_j^0, r'_j, r''_j) \to \mathbb{C}$,

$$f_{Z_j}(z_j) = f(z_j, Z_j) = f(z),$$

has a meromorphic extension to $B^2(z_j^0, r'_j)$. If $F$ is not contained in a countable union of analytic subvarieties in $G'$ of codimension $\geq 1$, then $f$ has a meromorphic extension to the open set

$$V(G', z_j^0, r'_j) = \{z = (z_j, Z_j) \in \mathbb{C}^n|z_j \in B^2(z_j^0, r'_j), Z_j \in G'\}.$$

Here, $B^2(z_0, r)$ is the disc in $\mathbb{C}$ with center $z_0$ and radius $r$, and $A(z_0, r_1, r_2)$ is the annulus $B^2(z_0, r_2) \setminus B^2(z_0, r_1)$. Following Siu [18, p. 17] we define the radius of meromorphy as follows (note that also other definitions are used, see for example [4, p. 578]). Let $f$ be a holomorphic function in the set $V(G', z_j^0, r'_j)$ where $G', j$, and $z_j^0$ are as above, and $r_j \in \mathbb{R}_+$. For each $Z_j \in G'$ the radius of meromorphy $\rho_j(Z_j)$ is the supremum of all $\rho > 0$ such that $f$ has a meromorphic extension to a neighborhood of the set

$$V(Z_j, z_j^0, \rho) = \{z = (z_j, Z_j) \in \mathbb{C}^n|z_j \in B^2(z_j^0, \rho)\}.$$
With this notation, we have

**Lemma 3** [18, Proposition 1.4 and Remark 1.5, pp. 17–18]. The function \( v_j : G' \to [-\infty, \infty), v_j(Z_j) = -\log p_j(Z_j), \) is subharmonic.

5. If \( f \) is a holomorphic function in \( G \setminus E \) and has a meromorphic extension \( f^* \) to \( G \), then the measure \( \Delta \log^+ |f| \) has locally finite mass near \( E \). This follows from the fact that each point \( z_0 \in G \) has a neighborhood \( U \) in \( G \) such that \( \log^+ |f^*| \) has a pluriharmonic majorant in \( U \setminus N(f^*) \). Here, \( N(f^*) \) is the nonsmooth set of \( f^* \); see, for example, [19, pp. 184–185]. As for the converse, we have

**Theorem 1.** Let \( E \) be \( n \)-small. Let \( f \) be a holomorphic function in \( G \setminus E \). If \( \Delta \log^+ |f| \) has locally finite mass near \( E \), then \( f \) has a meromorphic extension to \( G \).

**Proof.** If \( n = 1 \), then \( E \) is polar. Thus \( \log^+ |f| \) has locally a harmonic majorant near \( E \), and the theorem follows from [12, Théorème 20, p. 182].

Suppose then \( n \geq 2 \). It is sufficient to show that each point \( z^* \in E \) has a neighborhood \( V \) such that \( f|V \setminus E \) has a meromorphic extension to \( V \). As for the notation not explained in the sequel, see above or [7].

Take \( z^* \in E \) arbitrarily and choose \( r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+ \) such that

\[
U = D^n(z^*, 2r) = B^2(z_1^*, 2r_1) \times \cdots \times B^2(z_n^*, 2r_n) \subseteq G.
\]

For shortness, we write \( u = \log^+ |f| \) on \( U \setminus E \). Since \( \Delta u(U \setminus E) < \infty \), one sees, proceeding as Cegrell [1, proof of Theorem, pp. 284–285], using a nondecreasing sequence of nonnegative testfunctions in \( U \setminus E \) tending to \( 1 \), using the \( n \)-subharmonicity of \( u \), Fubini’s theorem, and the Monotone convergence theorem, that for each \( j, 1 \leq j \leq n \), there is a set \( B_j \subset \mathbb{C}^{n-1} \) such that \( H_{2n-2}(B_j) = 0 \) and that for each \( Z_j \in U(z_j^*) \setminus B_j \) the measure \( \Delta u_{Z_j} \) has locally finite mass near the section \( (E \cap U)(Z_j) \). For a detailed discussion of this, see [7, proof of Theorem 3.4, pp. 477–478]. Since \( E \) is \( n \)-small, by Lemma 1 we may suppose that the section \( (E \cap U)(Z_j) \) is polar in \( C \) for each \( Z_j \in U(z_j^*) \setminus B_j \).

Above we have used the notation

\[
U(z_j^*) = \{ Z_j \in \mathbb{C}^{n-1} | z = (z_j^*, Z_j) \in U \}, \quad (E \cap U)(Z_j) = \{ z_j \in \mathbb{C} | z = (z_j, Z_j) \in U \cap E \}
\]

for the sections of \( U \) in \( \mathbb{C}^{n-1} \) and of \( E \cap U \) in \( C \), respectively. Take \( Z_j^* \in U(z_j^*) \setminus B_j \) arbitrarily. Since the section \( (E \cap U)(Z_j^*) \) is then polar in \( C \) and \( E \) is closed in \( G \), there are \( r_j', r_j'', 0 < r_j' < r_j'' < 2r_j \) and a (connected) neighborhood \( W \subset U(z_j^*) \) of \( Z_j^* \) such that \( V(W, z_j^*, r_j', r_j'') \subseteq G \setminus E \). Since for each \( Z_j \in W \setminus B_j \) the measure \( \Delta u_{Z_j} \) has locally finite mass near the polar set \( (E \cap U)(Z_j) \), as shown above, we see as in the case \( n = 1 \) treated above, that the holomorphic functions \( f_{Z_j}|B^2(z_j, 2r_j)(E \cap U)(Z_j), Z_j \in W \setminus B_j \), have meromorphic extensions to \( B^2(z_j^*, 2r_j) \). Invoking this and the facts that \( f \) is holomorphic in \( V(W, z_j^*, r_j', r_j'') \) and \( H_{2n-2}(B_j) = 0 \), we see by Lemma 2 that \( f \) has a meromorphic extension to the set \( V(W, z_j^*, r_j') \). Since \( r_j'' \) can be chosen arbitrarily near to \( 2r_j' \), we have shown the following: For each \( j, 1 \leq j \leq n, \) and \( Z_j \in U(z_j^*) \setminus B_j \), the function \( f \) has a meromorphic extension to a neighborhood of the set \( V(Z_j, z_j^*, 2r_j) \). We refer to this result as condition (B).

Take \( z_0 \in D^n(z^*, r) \) and \( r = (r_0^1, \ldots, r_0^n) \in \mathbb{R}^n_+ \) such that \( V_0 = D^n(z_0, r) \subseteq D^n(z^*, r) \setminus E \). By (B) we see that \( \rho_1(Z_1) \geq r_1 \) for each \( Z_1 \in V_0(z_0) \setminus B_1 \). Thus
by Lemma 3 and, for example, by [6, Proposition 2b'), p. 10], \( \rho_1(Z_1) \geq r_1 \) for all \( Z_1 \in V_0(z_0^0) \), the section of \( V_0 \). Thus \( f \) has a meromorphic extension \( f_1 \) to \( V_1 = D^n(z_0, r^1) \), where \( r^1 = (r_1, r_2^0, \ldots, r_n^0) \). Observe that \( z_j^1 \in B^2(z_j^0, r_j^0) \). For the induction step suppose that \( 1 \leq k < n \) and \( r^k = (r_1, \ldots, r_k, r_{k+1}^0, \ldots, r_n^0) \in R^k_+ \), and \( f_k \) is a holomorphic extension of \( f \) to \( V_k = D^n(z_0, r^k) \). Observe that \( z_j^k \in B^2(z_j^0, r_j^0), j = 1, \ldots, k \). As above, using (B) we see that \( \rho_{k+1}(Z_{k+1}) \geq r_{k+1} \) for all \( Z_{k+1} \in V_k(z_{k+1}^0) \). Thus \( f_k \) and hence also \( f \) has a meromorphic extension \( f_{k+1} \) to \( V_{k+1} = D^n(z_0, r^{k+1}) \), where \( r^{k+1} = (r_1, \ldots, r_k, r_{k+1}^0, \ldots, r_n^0) \). Moreover, \( z_j^* \in B^2(z_j^0, r_j^0), j = 1, \ldots, k + 1, \) and \( V_{k+1} \subset U \). Thus the induction step is complete. Since \( V_n = D^n(z_0, r^n) \) is a neighborhood of \( z^* \), the proof is finished.

REMARK. The proof can also be based on [4, Theorem 2.9, p. 578] instead of Lemmas 2 and 3.

COROLLARY 1 [7, THEOREM 3.4, p. 477]. Let \( E \) be polar (respectively \( n \)-small). Let \( f \) be a holomorphic function in \( G \setminus E \). If \( \log^+ |f| \) has a harmonic majorant (respectively \( n \)-superharmonic majorant which is \( \not\equiv \infty \) on each component of \( G \setminus E \)), then \( f \) has a meromorphic extension to \( G \).

PROOF. By [6, Theorem 2, p. 25] (respectively [14, Theorem 4.1, p. 105]) the harmonic (respectively \( n \)-superharmonic) majorant \( h \) to \( \log^+ |f| \) has a superharmonic extension \( h^* \) to \( G \). Similarly, the subharmonic (respectively \( n \)-subharmonic) function \( \log^+ |f| - h \) has a subharmonic extension \( h_1^* \) to \( G \). Since \( \log^+ |f| = h^* + h_1^* \) in \( G \setminus E \) and \( -\Delta h^* \) and \( \Delta h_1^* \) are measures in \( G \), the corollary follows.

6. Computing the Laplacian, using Kametani's extension result [11, Theorem 2, p. 10, and Remark, p. 11] and proceeding as in the proof of Theorem 1 above, the following result will be proved in [8, Corollary 3.6, p. 301]: Let \( H_{2n-1}(E) = 0 \). Let \( f \) be a holomorphic function in \( G \setminus E \). If for some \( p > 0 \) the measure \( \Delta |f|^p \) has locally finite mass near \( E \), then \( f \) has a holomorphic extension to \( G \). The proof works also for measures \( \Delta (\log^+ |f|)^p, p > 1 \) (but not if \( p = 1 \)), instead of the measures \( \Delta |f|^p, p > 0 \). In fact, one can easily give a unified result which contains both cases:

THEOREM 2 (WITH J. HYVÖNEN). Let \( H_{2n-1}(E) = 0 \). Let \( f \) be a holomorphic function in \( G \setminus E \). Let \( \varphi: [-\infty, \infty) \to \mathbb{R} \) be a nondecreasing function such that \( \varphi|_{[\rho, \infty)} \) is strongly convex. Suppose that \( \varphi|_{(\rho, \infty)} \) is twice continuously differentiable for some \( \rho \in \mathbb{R} \). If the measure \( \Delta (\varphi \circ \log |f|) \) has locally finite mass near \( E \), then \( f \) has a holomorphic extension to \( G \).

OUTLINE OF PROOF. In view of the proof of Theorem 1 one may restrict to the case \( n = 1 \) (instead of the radius of meromorphy one can as well consider the radius of holomorphy). Take an open set \( D \subseteq G \) and set \( U = \{ z \in D \setminus E | \log |f(z)| > \rho \} \), where \( \rho \in \mathbb{R} \) is as in the theorem. Using a well-known inequality, computing the Laplacian and using the fact that \( \Delta (\varphi \circ \log |f|)(U) \leq \Delta (\varphi \circ \log |f|)(D \setminus E) < \infty \), one gets

\[
\int_{f(U)} \varphi''(\log |w|) \frac{1}{|w|^2} dm_2(w) \leq \int_{U} \varphi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dm_2(z) = \Delta (\varphi \circ \log |f|)(U) < \infty.
\]
On the other hand, since $\varphi$ is strongly convex,
\[
\int_{C \setminus B^2(0, r_0)} \varphi''(\log |w|) \frac{1}{|w|^2} \, dm_2(w) = 2\pi \int_{r_0}^{\infty} \varphi''(\log r) \frac{1}{r} \, dr = \infty
\]
for each $r_0 > e^p$. Thus $f$ omits a set of positive measure and has by the cited result of Kametani a meromorphic extension to $D$. Since this extension must be holomorphic, the theorem follows.

REMARK. Proceeding as in [8] one can also consider more general exceptional sets.

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