SUMMATION METHODS AND UNIQUENESS
IN VILENKIN GROUPS

D. J. GRUBB

(Communicated by Richard R. Goldberg)

ABSTRACT. It is shown that the $\sigma$-compact U-sets for two different summation methods on Vilenkin groups are the same.

Let $G$ be a Vilenkin group, i.e. a compact, totally disconnected, abelian, metric group, and let $\{H_n\}_{n=0}^{\infty}$ be a decreasing sequence of clopen subgroups of $G$ forming a neighborhood base at the identity of $G$. Let $\Gamma$ be the Pontryagin dual of $G$ and let $K_n = \text{Ann}(\Gamma, H_n)$ be the annihilator of $H_n$ in $\Gamma$. Then $\{K_n\}_{n=0}^{\infty}$ is an increasing sequence of finite subgroups of $\Gamma$ whose union is all of $\Gamma$.

Let $A(G)$ be the algebra of absolutely convergent Fourier series, $PM(G)$ the space of pseudomeasures, and $PF(G)$ the space of pseudofunctions, on $G$. Then $A(G) \simeq l_1(\Gamma)$, $PM(G) \simeq l_\infty(\Gamma)$, and $PF(G) \simeq c_0(\Gamma)$, where the isomorphisms are all written as $S \mapsto \hat{S}$.

If $S \in PM(G)$, we may define the $n$th partial sum of the Fourier series of type 1 of $S$ at a point $x$ in $G$ by

$$s_n(S,x) = s_n(S)(x) = \langle S, \xi_xH_n \rangle / \lambda(H_n) = \sum_{\gamma \in K_n} \hat{S}(\gamma) \gamma(x),$$

where $(S,f)$ realizes the duality $PM(G) \simeq A(G)^*$ of Banach spaces. This summation method is investigated in [1 and 2].

In [3] Vilenkin showed how to enumerate $\Gamma$ as $\{\gamma_n\}_{n=0}^{\infty}$ in such a way that for fixed $m \geq 0$, the sequence $\{\gamma_n\}_{n=0}^{\infty}$ fills cosets of $K_m$ successively. Also $\gamma_0 = 1$.

Using this enumeration, we may construct another summation method for trigonometric series. Simply investigate the series

$$\sum_{n=0}^{\infty} \hat{S}(\gamma_n) \gamma_n(x)$$

for $S \in PM(G)$. We call partial sums of this series partial sums of type 2, and this the Fourier series of $S$ of type 2.

We call a subset $E$ of $G$ a set of uniqueness or U-set of type 1 (resp. type 2) if the only Fourier series of type 1 (resp. type 2) of a pseudofunction converging to 0 everywhere except, possibly, on $E$ is the zero series. Notice that pseudofunctions are used in this definition and not pseudomeasures. This is important since otherwise there would be no U-sets of type 1. (See [1] for a discussion.)

Received by the editors December 2, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 43A55; Secondary 42C25.

Key words and phrases. Set of uniqueness, Vilenkin group.
The object of this paper is a demonstration that for $\sigma$-compact subsets of $G$, the $U$-sets of type 1 are exactly the $U$-sets of type 2. Because a partial sum of type 1 is a partial sum of type 2 (recall (*) and $\gamma_0 = 1$), a $U$-set of type 1 is a $U$-set of type 2, even if not $\sigma$-compact. It is also easy to show that the empty set is a $U$-set of type 1, and it is known that countable unions of closed $U$-sets of type 1 are again $U$-sets of type 1. (See [1 and 2].) It is also known that closed $U$-sets of type 1 are exactly those closed sets which support no pseudofunction in the distributional sense [2]. Thus three different definitions of the concept of a $U$-set coincide for closed sets.

A key step in our program is the following proposition. For the case when $G = \Pi \mathbb{Z}/(2)$, see [4 and 5].

**Theorem 1.** Let $S \in PF(G)$ and let $xH_m$ be a basic open set in $G$. Then the type 1 Fourier series of $S$ converges to 0 on $xH_m$ if and only if the type 2 Fourier series of $S$ converges to 0 on $xH_m$.

**Proof.** Since type 1 partial sums are type 2 partial sums, one direction is trivial.

Now assume that $s_n(S, y)$ converges to 0 for every $y$ in $xH_m$. Let $Q$ be a representative set in $\Gamma$ for the cosets of $K_m$, where $K_m$ is the annihilator of $H_m$. Then for $y$ in $H_m$ and $n \geq m$, write

$$s_n(S, xy) = \sum_{\gamma \in K_n} S(\gamma)(xy) = \sum_{\phi \in Q \cap K_n} \sum_{\eta \in K_m} S(\phi \eta)(\phi \eta)(xy)$$

$$= \sum_{\phi \in Q \cap K_n} \left[ \sum_{\eta \in K_m} S(\phi \eta)(\eta)(x) \right] \phi(y).$$

Since the Pontryagin dual of $H_m$ is isomorphic to $\Gamma/K_m$, this last sum may be regarded as a partial sum of a type 1 Fourier series on $H_m$. Since $s_n(S, xy)$ converges to 0 for all $y \in H_m$, and since the empty set is a $U$-set of type 1 in $H_m$, we get

$$\sum_{\eta \in K_m} S(\phi \eta)(\eta)(x) = 0 \quad \text{for } \phi \in Q.$$

If $y \in xH_m$, $yH_m = xH_m$, so the above holds with $y$ in place of $x$. Rewriting gives (1)

$$\sum_{\gamma \in \phi K_m} \hat{S}(\gamma)(\gamma)(y) = 0 \quad \text{for } y \in xH_m \text{ and } \phi \in Q.$$

Now, if $n \geq 0$ is given, then $\{\gamma_0, \ldots, \gamma_n\}$ is a union of cosets of $K_m$ except for a "tail" $\{\gamma_p, \ldots, \gamma_m\}$ lying entirely in one coset of $K_m$, since the sequence $\{\gamma_j\}_{j=1}^\infty$ fills up cosets of $K_m$ successively. This fact, with (1) shows

$$\left| \sum_{k=0}^n \hat{S}(\gamma_k)(\gamma_k)(y) \right| = \left| \sum_{k=p}^n \hat{S}(\gamma_k)(\gamma_k)(y) \right| \leq (n - p + 1) \sup |\hat{S}(\gamma)|$$

$$\leq \text{card}(K_m) \sup |\hat{S}(\gamma)| \to 0 \quad \text{for } y \in xH_m,$$

where the supremum is over a coset of $K_m$ which goes to infinity in $\Gamma$ as $n$ goes to infinity. This completes the proof of the proposition.
COROLLARY 2. A \( \sigma \)-compact subset of \( G \) is a \( U \)-set of type 1 if and only if it is a \( U \)-set of type 2.

PROOF. As noted above, any \( U \)-set of type 1 is a \( U \)-set of type 2 automatically. The theorem shows that a closed \( U \)-set of type 2 is a \( U \)-set of type 1. Now, if \( E = \bigcup_{n=0}^{\infty} E_n \) is a \( \sigma \)-compact \( U \)-set of type 2, with each \( E_n \) closed in \( G \), then each \( E_n \) is a \( U \)-set of type 2 and thus of type 1. Since countable unions of \( U \)-sets of type 1 are \( U \)-sets of type 1 (see [1 or 2]), \( E \) is a \( U \)-set of type 1.

BIBLIOGRAPHY

2. , Completeness and uniqueness on compact, 0-dimensional, metric spaces, unpublished.
5. V. A. Skvorcov, Example of a Walsh series with a subsequence of partial sums converging everywhere to 0, Mat. Sb. 97 (1975), 517–539.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115