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INVARIANT SUBSPACES FOR ALGEBRAS OF LINEAR OPERATORS AND AMENABLE LOCALLY COMPACT GROUPS

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ABSTRACT. Let G be a locally compact group. We prove in this paper that G is amenable if and only if the group algebra $L_1(G)$ (respectively the measure algebra $M(G)$) satisfies a finite-dimensional invariant subspace property $T(n)$ for n-dimensional subspaces contained in a subset X of a separated locally convex space E when $L_1(G)$ (respectively $M(G)$) is represented as continuous linear operators on E. We also prove that for any locally compact group, the Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

1. Introduction. Let E be a separated locally convex space and X a subset of E containing an n-dimensional subspace. In [**4**], K. Fan obtained the following finite-dimensional invariant subspace property $P(n)$ for n-dimensional subspaces contained in X: If $\mathcal{S} = \{T_s : s \in S\}$ is a representation of a left amenable (discrete) semigroup S as continuous linear operators from E into E such that $T_s(L)$ is an n-dimensional subspace contained in X whenever L is one and $s \in S$, and there exists a closed $\mathcal{S}$-invariant subspace H in E of codimension n with the property that $(x + H) \cap X$ is compact convex for each $x \in E$, then there exists an n-dimensional subspace L_0 contained in X such that $T_s(L_0) = L_0$ for all $s \in S$. Conversely, as proved by Lau [**10**] (see also [**12**]), if S satisfies $P(1)$, then S is left amenable.

In this paper, we prove among other things that (Theorem 3.5) if G is an amenable locally compact group, then the group algebra $L_1(G)$ and measure algebra $M(G)$ satisfy a similar finite-dimensional invariant subspace property $T(n)$ for n-dimensional subspaces contained in a subset X of a separated locally convex subspace E when $L_1(G)$ (or $M(G)$) is represented as continuous linear operators on E. Conversely, if $L_1(G)$ or $M(G)$ satisfies $T(1)$, then G is amenable. We also show that (Theorem 3.6) if G in *any* locally compact group, then both the Fourier algebra $A(G)$ and Fourier Stieltjes algebra $B(G)$ (see Eymard [**3**]) always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

For convenience and to avoid repetition of arguments, we find it most natural to phrase our main result (Theorem 3.4) in terms of F-algebras (see [**11**]). Definition and information on F-algebras needed for the proof of our main result will be gathered in §2.

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2. F-algebras. By an *F-algebra* we shall mean a complex Banach algebra A such that A^* is a C^*-algebra and the identity of A^*, denoted by I (which always exists [**15**, Proposition 1.6.1]), is a multiplicative linear functional on A. Examples of F-algebras include the group algebra $L_1(G)$, Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ of a locally compact group G (see [**11**] for details). It also includes the measure algebra $M(S)$ of a locally compact semigroup S.

Let S_A denote the set of all positive functionals in $A \subset A^{**}$ with norm one. Then $S_A = \{\mu \in A; \|\mu\| = I(\mu) = 1\}$ [**15**, p. 9]. Hence, as readily checked, $(S_A, *)$, where $*$ denotes the multiplication of A, is a semigroup. A is called *left amenable* if $A^* = M$ has a topological left invariant mean (abbreviated as TLIM), i.e. an $m \in M^*$ such that $\|m\| = 1$, $m \geq 0$ and $m(F \cdot \mu) = m(F)$ for each $\mu \in S_A$ and $F \in M$, where $F \cdot \mu \in M$ is defined by $\langle F \cdot \mu, \nu\rangle = \langle F, \mu * \nu\rangle$ for all $\nu \in A$.

A function $f \in \mathrm{CB}(S_A)$ (the continuous bounded functions on S_A), is called additively uniformly continuous on S_A if given $\varepsilon > 0$, there is some $\delta > 0$ such that $\mu, \nu \in S_A$ and $\|\mu - \nu\| < \varepsilon$ implies $|f(\mu) - f(\nu)| < \varepsilon$. Let $\mathrm{AUC}(S_A)$ denote the space of all such functions. It is straightforward to show that $\mathrm{AUC}(S_A)$ is a norm closed translation invariant subspace of $\mathrm{CB}(S_A)$ containing constants and restrictions of elements in A^* to S_A. A linear functional $m \in \mathrm{AUC}(S_A)^*$ is called a left invariant mean if $\|m\| = m(1) = 1$ and $m(l_\mu f) = m(f)$, $\forall f \in \mathrm{AUC}(S_A)$, $\mu \in S_A$, where $(l_\mu f)(\nu) = f(\mu * \nu)$ for all $\nu \in S_A$. The following lemma, which we shall need, is due to Ganeson [**5**] for the case $A = L_1(G)$ of a locally compact group. Because part of his proof depends heavily on the locally compact group structure, we include a simple proof here for completeness.

LEMMA 2.1. *The following statements are equivalent for an F-algebra A*:
(a) *A is left amenable.*
(b) $\mathrm{AUC}(S_A)$ *has a left invariant mean.*

PROOF. (a) *implies* (b). By [**11**, Theorem 4.6], there is a net $\mu_\alpha \in S_A$ such that $\mu * \mu_\alpha - \mu_\alpha \to 0$ in the norm topology for each $\mu \in S_A$. For each α, define $m_\alpha(f) = f(\mu_\alpha)$ for each $f \in \mathrm{AUC}(S_A)$. Let m be a weak*-cluster point of $\{m_\alpha\}$. Then m is a left invariant mean on $\mathrm{AUC}(S_A)$.

(b) *implies* (a). Define $\tau\colon A^* \to \mathrm{CB}(S_A)$ by $\tau(F)(\mu) = F(\mu)$, $F \in A^*$, $\mu \in S_A$. Then τ is a continuous linear map of A^* into $\mathrm{CB}(S_A)$ such that $\tau \geq 0$, $\tau(1) = 1$ and $\tau(F \cdot \mu) = l_\mu(\tau(F))$ where $\mu \in S_A$, $F \in A^*$ and l_μ is the left translation operator in $\mathrm{CB}(S_A)$. Moreover, $\tau(F) \in \mathrm{AUC}(S_A)$. Hence $\tau^*\colon \mathrm{AUC}(S_A)^* \to A^{**}$. If m is a left invariant mean on $\mathrm{AUC}(S_A)$, then $\tau^*(m)$ is a TLIM on A^*.

3. Algebra of operators. A representation of an F-algebra A as operators in a locally convex space E is a map $T\colon A \times E \to E$ denoted by $(\mu, x) \to T_\mu(x)$ such that (1) $T_\mu\colon E \to E$ is continuous and linear, (2) $\mu \to T_\mu(x)$ is continuous and linear with respect to the norm topology in A for each $x \in E$ and (3) $T_{\mu * \nu} = T_\mu \circ T_\nu$ $\forall \mu, \nu \in A$, where $*$ denotes multiplication in A. Also, let X be a subset of E containing an n-dimensional subspace. As in Lau [**10**], $\mathcal{L}_n(X)$ denotes all n-dimensional subspaces of E contained in X. We say that $\mathcal{L}_n(X)$ is S_A-invariant under T if $T_\mu(L) \in \mathcal{L}_n(X)$ for each $L \in \mathcal{L}_n(X)$ and $\mu \in S_A$. A closed subspace H in E is called S_A-invariant under T if $T_\mu(H) \subset H$ $\forall \mu \in S_A$ (and hence $\forall \mu \in A$ as well). Denote by $q\colon E \to E/H$ the natural map such that $q(x) = \tilde{x}$, $x \in E$.

LEMMA 3.1. *Let $T\colon A \times E \to E$ be a representation of an F-algebra A and H a closed S_A-invariant subspace of E of codimension n. Also let X be a subset of E such that $(x+H) \cap X$ is compact and convex for each $x \in E$. Denote by $\mathcal{K}$ the set of all linear maps $B \in \mathcal{L}(F,E)$ such that $B(y) \in q^{-1}(y) \cap X$ $\forall y \in F$. If $\mathcal{L}_n(X)$ is nonempty and S_A-invariant, then for each $\mu \in A$, the map $T_\mu\colon E \to E$ induces a map $\tilde{T}_\mu\colon F \to F$ where $F = E/H$ such that $q \circ T_\mu = \tilde{T}_\mu \circ q$ $\forall \mu \in A$. Moreover, if $\mu \in S_A$, $\tilde{T}_\mu$ is an isomorphism of F onto itself and $\mathcal{K} \neq \emptyset$ is convex and compact in the separated locally convex space $\mathcal{L}(F,E)$ with the topology τ of pointwise convergence. Defining $\psi\colon S_A \times \mathcal{K} \to \mathcal{K}$ by $\psi_\mu(B) = T_\mu \circ B \circ \tilde{T}_\mu^{-1}$, $\mu \in S_A$, $B \in \mathcal{K}$, then ψ is an affine action of the semigroup S_A on $\mathcal{K}$.*

PROOF. The proof is basically contained in Fan [**4**, Theorem, p. 447]. We need only consider the semigroup S_A of linear operators in E. Note that $\tilde{T}_\mu$ is defined for $\mu \in A$ (not just in S_A). However $\tilde{T}_\mu$ need not be an isomorphism unless $\mu \in S_A$.

DEFINITION 3.2. The action of S_A on E is called *inversely equicontinuous modulo H* if given any neighborhood U in E, there is some neighborhood V in E such that $V \subset T_\mu(U) + H$ for any $\mu \in S_A$. This is equivalent to the condition that the family $\{\tilde{T}_\mu^{-1}\colon \mu \in S_A\}$ is equicontinuous on F.

LEMMA 3.3. *Under the hypothesis of Lemma* 3.1 *and if A is left amenable and the action of S_A on E is inversely equicontinuous modulo H, then the induced action $\tilde{T}\colon S_A \times F \to F$, where $(\mu, y) \to \tilde{T}_\mu y$, $\mu \in S_A$ and $y \in F$, is similar to a unitary representation of the semigroup S_A on the Hilbert space F. Moreover, the representation functions $\mu \to (\tilde{T}_\mu y, z)$, $y, z \in F$, are in* AUC(S_A). *The same is ture for the inverse* (*anti*)*representation $\{\tilde{T}_\mu^{-1}\colon \mu \in S_A\}$.*

PROOF. Fix a basis $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ in F and define $(y,z) = \sum_{j=1}^n \alpha_j \overline{\beta}_j$ where $y = \sum_{j=1}^n \alpha_j \tilde{e}_j$ and $z = \sum_{j=1}^n \beta_j \tilde{e}_j$ are in F, and let $\|\cdot\|$ denote the induced (Euclidean) norm. Consider the adjoint (anti)representation of S_A on F, i.e. $\tilde{T}^*\colon S_A \times F \to F$ such that $(\mu, y) \to \tilde{T}_\mu^* y$ where $\mu \in S_A$, $y \in F$ and $\tilde{T}_\mu^*$ is the adjoint of $\tilde{T}_\mu$ on the Hilbert space F. Since the map $\mu \to \tilde{T}_\mu^* y$ is continuous and conjugate linear on A and since

$$| \, \|\tilde{T}_\mu^* y\| - \|\tilde{T}_\nu^* y\| \, | \le \|\tilde{T}_\mu^* y - \tilde{T}_\nu^* y\| \le \|\mu - \nu\| \cdot M(y) \quad \forall \mu, \nu \in S_A,$$

where $M(y) \ge 0$, it follows that the function $\mu \to \|\tilde{T}_\mu^* y\|$ and hence $\mu \to \|\tilde{T}_\mu^* y\|^2$ on S_A is in AUC(S_A). By the Polarisation Principle, the function $\mu \to (\tilde{T}_\mu^* y, \tilde{T}_\mu^* z)$ (restricted to S_A) is also in AUC(S_A). Let m be a left invariant mean on AUC(S_A) (by Lemma 2.1). Define a new inner product on F by $[y,z] = \langle m(\mu), (\tilde{T}_\mu^* y, \tilde{T}_\mu^* z)\rangle$. Now the family $\{\tilde{T}_\mu^*\colon \mu \in S_A\}$ is pointwise bounded on F, hence bounded on F (by the Principle of Uniform Boundedness). On the other hand, inverse equicontinuity of S_A modulo H implies the family $\{\tilde{T}_\mu^{-1}\colon \mu \in S_A\}$ is equicontinuous hence uniformly bounded on F. So is the family $\{(\tilde{T}_\mu^*)^{-1}\colon \mu \in S_A\} = \{(\tilde{T}_\mu^{-1})^*\colon \mu \in S_A\}$. Therefore there exist $M, N > 0$ such that

$$\begin{aligned} N^2\|y\|^2 &\le \inf\{\|\tilde{T}_\mu^*(y)\|^2\colon \mu \in S_A\} \le \langle m(\mu), \|\tilde{T}_\mu^*(y)\|^2\rangle \\ &= |y|^2 \le \sup\{\|\tilde{T}_\mu^*(y)\|^2\colon \mu \in S_A\} \le M^2\|y\|^2, \end{aligned}$$

where $|y|^2 = [y,y]$. Consequently, the two norms are equivalent and hence, as is well known (see [**7**]), there is an invertible bicontinuous selfadjoint linear operator $Q \in \mathcal{B}(F)$ such that $[y,z] = (Qy, Qz)$ $\forall y, z \in F$. Now $\forall \mu \in S_A$,

$$\begin{aligned}\|Q\tilde{T}_\mu^* Q^{-1}(y)\|^2 &= |\tilde{T}_\mu^* Q^{-1}(y)|^2 = \langle m(\nu), \|\tilde{T}_\nu^* \tilde{T}_\mu^* Q^{-1}(y)\|^2\rangle \\ &= \langle m(\nu), \|\tilde{T}_{\mu*\nu}^* Q^{-1} y)\|^2\rangle = \langle m(\nu), \|\tilde{T}_\nu^* Q^{-1}(y)\|^2\rangle \\ &= |Q^{-1}(y)|^2 = \|y\|^2\end{aligned}$$

by left invariance of m. Therefore $U_\mu = Q\tilde{T}_\mu^* Q^{-1}$ is unitary for any $\mu \in S_A$. This implies that $\{\tilde{T}_\mu : \mu \in S_A\}$ is similar to a unitary representation on F. Finally, since $\mu \to (\tilde{T}_\mu y, z)$ is bounded linear on A, its restriction to S_A is in $\mathrm{AUC}(S_A)$. To show that the same is true for the inverse representation, we first observe that since F is finite dimensional, the strong operator topology and the uniform operator topology agree on $\mathcal{B}(F)$. Hence $\mu \to \tilde{T}_\mu^*$ is also continuous when $B(F)$ has the uniform operator topology. Consequently given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu, \nu \in A$, $\|\mu - \nu\| < \delta$ implies $\|\tilde{T}_\mu^* - \tilde{T}_\nu^*\| < \varepsilon$. In particular, if $\mu, \nu \in S_A$ and $\|\mu - \nu\| < \delta$, then $\|U_\mu - U_\nu\| = \|Q\tilde{T}_\mu^* Q^{-1} - Q\tilde{T}_\nu^* Q^{-1}\| \le \|Q\| \cdot \|Q^{-1}\| \cdot \varepsilon$. But $\tilde{T}_\mu^{-1} = QU_\mu Q^{-1}$ $\forall \mu \in S_A$, since U_μ is unitary for such μ and Q is selfadjoint. Therefore

$$\|\tilde{T}_\mu^{-1} - \tilde{T}_\nu^{-1}\| \le \|Q\|^2 \cdot \|Q^{-1}\|^2 \cdot \varepsilon \quad \forall \mu, \nu \in S_A$$

with $\|\mu - \nu\| < \delta$. This implies that the function $\mu \to y^* \tilde{T}_\mu^{-1} y$ on S_A is in $\mathrm{AUC}(S_A)$ $\forall y \in F$, $y^* \in F^*$.

THEOREM 3.4. *Let A be an F-algebra.*

(a) *If A is left amenable, then A satisfies property $T(n)$ for $n = 1, 2, \ldots$ where property $T(n)$ is defined as follows: Let E be a separated locally convex space and $T: A \times E \to E$ be a representation of A as linear operators in E. Let X be a subset of E such that there exists a closed S_A-invariant subspace H of E of codimension n and $(x + H) \cap X$ is compact convex for each $x \in E$. If the action of S_A on E is inversely equicontinuous modulo H and $\mathcal{L}_n(X)$ is nonempty and S_A-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_\mu(L_0) = L_0$ $\forall \mu \in S_A$.*

(b) *If S satisfies property $T(1)$, then A is left amenable (hence A satisfies $T(n)$ for every n).*

PROOF. (a) With the notation of Lemma 3.1, define the affine representaion $\Psi: S_1 \times \mathcal{K} \to \mathcal{K}$ by $\Psi_\mu(B) = T_\mu \circ B \circ \tilde{T}_\mu^{-1}$, $\mu \in S_A$ and $B \in \mathcal{K}$. We want to show that ψ is an A-representation for the pair S_A and $\mathrm{AUC}(S_A)$ in the sense of Argabright [**1**, §2]. That is, for each $h \in A(\mathcal{K})$, the affine continuous functions on $\mathcal{K}$, the function $\mu \to h(\Psi_\mu(B))$ is in $\mathrm{AUC}(S_A)$ for each $B \in \mathcal{K}$. By Argabright [**1**, Lemma 1] and Kelley and Namioka [**9**, Theorem 14.6, p. 120], it is enought to consider $h \in A(\mathcal{K})$ of the form $h(B) = x^* By$, $B \in \mathcal{K}$, where $x^* \in E^*$ and $y \in F$. Let $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ be a basis in F. Then $\{qT_\mu B\tilde{e}_1, \ldots, qT_\mu B\tilde{e}_n\}$ is also a basis in F for any $\mu \in S_A$ and $B \in \mathcal{K}$ (by definition of $\mathcal{K}$, S_A-invariance of $\mathcal{L}_n(X)$ and the fact that $(x + H) \cap X$ is compact and convex for each $x \in E$). Write $y = \sum_{j=1}^n \alpha_j(\mu) qT_\mu B(\tilde{e}_j)$ where $\mu \in S_A$, $B \in \mathcal{K}$. Note that the scalars $\alpha_j(\mu)$ depend only on $\mu \in S_A$ and not on B. (See Fan [**4**, equation (7), p. 449].) Then $\tilde{T}_\mu^{-1}(y) = \sum_{j=1}^n \alpha_j(\mu) \tilde{e}_j$, $\mu \in S_A$. (Since $q \circ B =$ identity on F.) By Lemma 3.3, the functions $\mu \to \alpha_k(\mu) = y_k^* \tilde{T}_\mu^{-1}(y)$ on

S_A belong to $\mathrm{AUC}(S_A)$ where $\{y_1^*, \ldots, y_n^*\}$ is the basis dual to $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. Now

$$h(\Psi_\mu(B)) = x^*(T_\mu \circ B \circ \tilde{T}_\mu^{-1})y = \sum_{j=1}^{n} \alpha_j(\mu) x^* T_\mu B(\tilde{e}_j), \qquad \mu \in S_A.$$

Therefore the function $\mu \to h(\Psi_\mu(B))$ on S_A is in $\mathrm{AUC}(S_A)$. Hence $\Psi\colon S_1 \times \mathcal{K} \to \mathcal{K}$ is an A-representation for the pair S_1, $\mathrm{AUC}(S_A)$. By Argabright [**1**, Theorem 1], Ψ has a common fixed point $Q_0 \in \mathcal{K}$. Put $L_0 = Q_0(F) \in \mathcal{L}_n(X)$, then $T_\mu(L_0) = L_0$ $\forall \mu \in S_A$.

(b) Define $E = A^{**}$ with the weak* topology and $T\colon A \times E \to E$ by $T_\mu(m) = l_\mu^* m$ where l_μ^* is the adjoint of the map $l_\mu\colon A^* \to A^*$ such that $l_\mu(F)(\nu) = F(\mu * \nu)$, $\nu \in A$. As in Lau [**10**], let X be the union of all one-dimensional subspaces of $E = A^{**}$ generated by the means (states) on A^* and $H = \{m \in A^{**} : m(I) = 0\}$. The arguments used in the proof of Lau [**7**, Theorem 1(b)] show that the hypotheses in property $T(1)$ are all satisfied except the part on inverse equicontinuity modulo H of S_A, which is also satisfied because the induced action $\tilde{T}_\mu$ is independent of $\mu \in S_A$. ($l_\mu(I) = I$ if $\mu \in S_A$, since $I(\mu) = 1$, if $\mu \in S_A$.) Since S satisfies $T(1)$, this means $l_\mu^*(L_0) = L_0$ $\forall \mu \in S_A$, where L_0 is a one-dimensional subspace generated by some mean m_0. Necessarily, m_0 is a topological left invariant mean on A^*.

A locally compact group G is *amenable* if the space $\mathrm{CB}(G)$ has a left invariant mean. Examples of amenable locally compact groups include all solvable groups, abelian groups and all compact groups. However, if G contains the free group on two generators as a closed subgroup, then G is not amenable (see Greenleaf [**7**] or Pier [**13**] for details).

THEOREM 3.5. *Let G be a locally compact group. If G is amenable, then the group algebra $L_1(G)$ and the measure algebra $M(G)$ satisfy $T(n)$ for each $n = 1, 2, 3, \ldots$. Conversely, if either $L_1(G)$ or $M(G)$ satisfies $T(1)$, then G is amenable.*

PROOF. It follows from Greenleaf [**7**, Theorem 2.2.1] and Wong [**17**, Theorem 3.3] that amenability of G is equivalent to the left amenability of $L_1(G)$ and $M(G)$.

THEOREM 3.6. *Let G be a locally compact group. Then both the Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ have property $T(n)$ for each $n = 1, 2, \ldots$.*

PROOF. In this case, if $A = A(G)$ or $B(G)$, the A is a commutative F-algebra (see Eymard [**3**]). Hence S_A is commutative and so $l_\infty(S_A)$, the space of bounded complex-valued functions on the discrete semigroup S_A, has an invariant mean (see Day [**2**]). So by Lemma 2.1, A is left amenable. Now apply Theorem 3.4.

Given a semigroup S, let $l_1(S)$ be the Banach algebra as defined in Day [**2**] or Hewitt and Zuckermann [**8**]. S is called left amenable if $l_\infty(S)$ has a left invariant mean (see Day [**2**]).

THEOREM 3.7. *Let S be a semigroup. If S is left amenable, then the Banach algebra $l_1(S)$ satisfies $T(n)$ for each $n = 1, 2, 3, \ldots$. Conversely if $l_1(S)$ satisfies $T(1)$, then S is left amenable.*

PROOF. Any left invariant mean on $l_\infty(S)$ is necessarily topological left invariant. Now apply Theorem 3.4 again.

REMARKS. (a) Left amenability of the Fourier algebra $A(G)$ of a locally compact group G was proved by Renauld [**14**, Theorem 4] (see also Granirer [**6**, Proposition 5(a)]) by a completely different method.

(b) Theorem 3.7 can be generalized to all locally compact semigroups S with left amenability of S replaced by topological left amenability of the measure algebra $M(S)$ (see [**16**]), and $l_1(S)$ replaced by $M(S)$.

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