IN Variant Subspaces  
for Algebras of Linear Operators  
and Amenable Locally Compact Groups  

ANTHONY T. M. LAU AND JAMES C. S. WONG  

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Abstract. Let $G$ be a locally compact group. We prove in this paper that $G$ is amenable if and only if the group algebra $L_1(G)$ (respectively the measure algebra $M(G)$) satisfies a finite-dimensional invariant subspace property $T(n)$ for $n$-dimensional subspaces contained in a subset $X$ of a separated locally convex space $E$ when $L_1(G)$ (respectively $M(G)$) is represented as continuous linear operators on $E$. We also prove that for any locally compact group, the Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

1. Introduction. Let $E$ be a separated locally convex space and $X$ a subset of $E$ containing an $n$-dimensional subspace. In [4], K. Fan obtained the following finite-dimensional invariant subspace property $P(n)$ for $n$-dimensional subspaces contained in $X$: If $S = \{T_s : s \in S\}$ is a representation of a left amenable (discrete) semigroup $S$ as continuous linear operators from $E$ into $E$ such that $T_s(L)$ is an $n$-dimensional subspace contained in $X$ whenever $L$ is one and $s \in S$, and there exists a closed $S$-invariant subspace $H$ in $E$ of codimension $n$ with the property that $(x + H) \cap X$ is compact convex for each $x \in E$, then there exists an $n$-dimensional subspace $L_0$ contained in $X$ such that $T_s(L_0) = L_0$ for all $s \in S$. Conversely, as proved by Lau [10] (see also [12]), if $S$ satisfies $P(1)$, then $S$ is left amenable.

In this paper, we prove among other things that (Theorem 3.5) if $G$ is an amenable locally compact group, then the group algebra $L_1(G)$ and measure algebra $M(G)$ satisfy a similar finite-dimensional invariant subspace property $T(n)$ for $n$-dimensional subspaces contained in a subset $X$ of a separated locally convex subspace $E$ when $L_1(G)$ (or $M(G)$) is represented as continuous linear operators on $E$. Conversely, if $L_1(G)$ or $M(G)$ satisfies $T(1)$, then $G$ is amenable. We also show that (Theorem 3.6) if $G$ in any locally compact group, then both the Fourier algebra $A(G)$ and Fourier Stieltjes algebra $B(G)$ (see Eymard [3]) always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

For convenience and to avoid repetition of arguments, we find it most natural to phrase our main result (Theorem 3.4) in terms of $F$-algebras (see [11]). Definition and information on $F$-algebras needed for the proof of our main result will be gathered in §2.

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581
2. F-algebras. By an F-algebra we shall mean a complex Banach algebra \( A \) such that \( A^* \) is a C*-algebra and the identity of \( A^* \), denoted by \( I \) (which always exists [15, Proposition 1.6.1]), is a multiplicative linear functional on \( A \). Examples of F-algebras include the group algebra \( L_1(G) \), Fourier algebra \( A(G) \) and the Fourier Stieltjes algebra \( B(G) \) of a locally compact group \( G \) (see [11] for details). It also includes the measure algebra \( M(S) \) of a locally compact semigroup \( S \).

Let \( S_A \) denote the set of all positive functionals in \( A \subset A^{**} \) with norm one. Then \( S_A = \{ \mu \in A; \| \mu \| = I(\mu) = 1 \} \) [15, p. 9]. Hence, as readily checked, \((S_A, \star)\), where \( \star \) denotes the multiplication of \( A \), is a semigroup. \( A \) is called left amenable if \( A^* = M \) has a topological left invariant mean (abbreviated as TLIM), i.e. an \( m \in M^* \) such that \( \|m\| = 1, m \geq 0 \) and \( m(F \cdot \mu) = m(F) \) for each \( \mu \in S_A \) and \( F \in M \), where \( F \cdot \mu \in M \) is defined by \( (F \cdot \mu, \nu) = (F, \mu \star \nu) \) for all \( \nu \in A \).

A function \( f \in CB(S_A) \) (the continuous bounded functions on \( S_A \)), is called additively uniformly continuous on \( S_A \) if given \( \varepsilon > 0 \), there is some \( \delta > 0 \) such that \( \mu, \nu \in S_A \) and \( \| \mu - \nu \| < \delta \) implies \( |f(\mu) - f(\nu)| < \varepsilon \). Let \( AUC(S_A) \) denote the space of all such functions. It is straightforward to show that \( AUC(S_A) \) is a norm closed translation invariant subspace of \( CB(S_A) \) containing constants and restrictions of elements in \( A^* \) to \( S_A \). A linear functional \( m \in AUC(S_A)^* \) is called a left invariant mean if \( \|m\| = m(1) = 1 \) and \( m(l_\mu f) = m(f) \), \( \forall f \in AUC(S_A), \mu \in S_A \), where \( (l_\mu f)(\nu) = f(\mu \star \nu) \) for all \( \nu \in S_A \). The following lemma, which we shall need, is due to Ganeson [5] for the case \( A = L_1(G) \) of a locally compact group. Because part of his proof depends heavily on the locally compact group structure, we include a simple proof here for completeness.

**Lemma 2.1.** The following statements are equivalent for an F-algebra \( A \):

(a) \( A \) is left amenable.

(b) \( AUC(S_A) \) has a left invariant mean.

**Proof.** (a) implies (b). By [11, Theorem 4.6], there is a net \( \mu_\alpha \in S_A \) such that \( \mu \star \mu_\alpha - \mu_\alpha \to 0 \) in the norm topology for each \( \mu \in S_A \). For each \( \alpha \), define \( m_\alpha(f) = f(\mu_\alpha) \) for each \( f \in AUC(S_A) \). Let \( m \) be a weak*-cluster point of \( \{m_\alpha\} \). Then \( m \) is a left invariant mean on \( AUC(S_A) \).

(b) implies (a). Define \( \tau: A^* \to CB(S_A) \) by \( \tau(F)(\mu) = F(\mu), F \in A^*, \mu \in S_A \). Then \( \tau \) is a continuous linear map of \( A^* \) into \( CB(S_A) \) such that \( \tau \geq 0, \tau(1) = 1 \) and \( \tau(F \cdot \mu) = l_\mu(\tau(F)) \) where \( \mu \in S_A, F \in A^* \) and \( l_\mu \) is the left translation operator in \( CB(S_A) \). Moreover, \( \tau(F) \in AUC(S_A) \). Hence \( \tau^*: AUC(S_A)^* \to A^{**} \). If \( m \) is a left invariant mean on \( AUC(S_A) \), then \( \tau^*(m) \) is a TLIM on \( A^* \).

3. Algebra of operators. A representation of an F-algebra \( A \) as operators in a locally convex space \( E \) is a map \( T: A \times E \to E \) denoted by \( (\mu, x) \to T_\mu(x) \) such that (1) \( T_\mu: E \to E \) is continuous and linear, (2) \( \mu \to T_\mu(x) \) is continuous and linear with respect to the norm topology in \( A \) for each \( x \in E \) and (3) \( T_{\mu \star \nu} = T_\mu \circ T_\nu \forall \mu, \nu \in A \), where \( \star \) denotes multiplication in \( A \). Also, let \( X \) be a subset of \( E \) containing an \( n \)-dimensional subspace. As in Lau [10], \( \mathcal{L}_n(X) \) denotes all \( n \)-dimensional subspaces of \( E \) contained in \( X \). We say that \( \mathcal{L}_n(X) \) is \( S_A \)-invariant under \( T \) if \( T_\mu(L) \in \mathcal{L}_n(X) \) for each \( L \in \mathcal{L}_n(X) \) and \( \mu \in S_A \). A closed subspace \( H \) in \( E \) is called \( S_A \)-invariant under \( T \) if \( T_\mu(H) \subset H \forall \mu \in S_A \) (and hence \( \forall \mu \in A \) as well). Denote by \( q: E \to E/H \) the natural map such that \( q(x) = \bar{x}, x \in E \).

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LEMMA 3.1. Let T: A x E — E be a representation of an F-algebra A and H a closed S^A-invariant subspace of E of codimension n. Also let X be a subset of E such that (x + H) \cap X is compact and convex for each x \in E. Denote by K the set of all linear maps B \in \mathcal{L}(F,E) such that B(y) = q^{-1}(y) \cap X \forall y \in F. If L_n(X) is nonempty and S^A-invariant, then for each \mu \in A, the map T_\mu: E — E induces a map \tilde{T}_\mu: F — F where F = E/H such that q \circ T_\mu = \tilde{T}_\mu \circ q \forall \mu \in A. Moreover, if \mu \in S^A, \tilde{T}_\mu is an isomorphism of F onto itself and K \neq \emptyset is convex and compact in the separated locally convex space \mathcal{L}(F,E) with the topology \tau of pointwise convergence. Defining \psi: S^A \times K \rightarrow K by \psi_\mu(B) = T_\mu \circ B \circ T_\mu^{-1}, \mu \in S^A, B \in K, then \psi is an affine action of the semigroup S^A on K.

PROOF. The proof is basically contained in Fan [4, Theorem, p. 447]. We need only consider the semigroup S^A of linear operators in E. Note that \tilde{T}_\mu is defined for \mu \in A (not just in S^A). However \tilde{T}_\mu need not be an isomorphism unless \mu \in S^A.

DEFINITION 3.2. The action of S^A on E is called inversely equicontinuous modulo H if given any neighborhood U in E, there is some neighborhood V in E such that V \subset T_\mu(U) + H for any \mu \in S^A. This is equivalent to the condition that the family \{\tilde{T}_\mu^{-1}: \mu \in S^A\} is equicontinuous on F.

LEMMA 3.3. Under the hypothesis of Lemma 3.1 and if A is left amenable and the action of S^A on E is inversely equicontinuous modulo H, then the induced action \tilde{T}: S^A \times F — F, where (\mu, y) \rightarrow \tilde{T}_\mu y, \mu \in S^A and y \in F, is similar to a unitary representation of the semigroup S^A on the Hilbert space F. Moreover, the representation functions \mu \rightarrow (\tilde{T}_\mu y, z), y, z \in F, are in AUC(S^A). The same is true for the inverse (anti)representation \{\tilde{T}_\mu^{-1}: \mu \in S^A\}.

PROOF. Fix a basis \{\tilde{e}_1, \ldots, \tilde{e}_n\} in F and define (y, z) = \sum_{j=1}^n \alpha_j \overline{\beta}_j where y = \sum_{j=1}^n \alpha_j \tilde{e}_j and z = \sum_{j=1}^n \beta_j \tilde{e}_j are in F, and let \|\cdot\| denote the induced (Euclidean) norm. Consider the adjoint (anti)representation of S^A on F, i.e. \tilde{T}^*: S^A \times F — F such that (\mu, y) \rightarrow \tilde{T}_\mu^* y where \mu \in S^A, y \in F and \tilde{T}_\mu^* is the adjoint of \tilde{T}_\mu on the Hilbert space F. Since the map \mu \rightarrow \tilde{T}_\mu y is continuous and conjugate linear on A and since

\[ \|\tilde{T}_\mu^* y \| \leq \|\tilde{T}_\mu^* y - \tilde{T}_\nu^* y \| \leq \|\mu - \nu\| \cdot M(y) \forall \mu, \nu \in S^A, \]

where M(y) \geq 0, it follows that the function \mu \rightarrow \|\tilde{T}_\mu^* y \| and hence \mu \rightarrow \|\tilde{T}_\mu^* y \|^2 on S^A is in AUC(S^A). By the Polarity Principle, the function \mu \rightarrow (\tilde{T}_\mu^* y, \tilde{T}_\mu^* z) (restricted to S^A) is also in AUC(S^A). Let m be a left invariant mean on AUC(S^A) (by Lemma 2.1). Define a new inner product on F by \langle y, z \rangle = \langle m(\mu), (\tilde{T}_\mu^* y, \tilde{T}_\mu^* z) \rangle.

Now the family \{\tilde{T}_\mu^*: \mu \in S^A\} is pointwise bounded on F, hence bounded on F (by the Principle of Uniform Boundedness). On the other hand, inverse equicontinuity of S^A modulo H implies the family \{\tilde{T}_\mu^{-1}: \mu \in S^A\} is equicontinuous hence uniformly bounded on F. So is the family \{(\tilde{T}_\mu^*)^{-1}: \mu \in S^A\} = \{\tilde{T}_\mu^{-1}: \mu \in S^A\}. Therefore there exist M, N > 0 such that

\[ \inf \{\|\tilde{T}_\mu^*(y)\|^2: \mu \in S^A\} \leq \langle m(\mu), \|\tilde{T}_\mu^*(y)\|^2 \rangle \]

\[ = \|y\|^2 \leq \sup \{\|\tilde{T}_\mu^*(y)\|^2: \mu \in S^A\} \leq M^2 \|y\|^2, \]

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where $|y|^2 = [y, y]$. Consequently, the two norms are equivalent and hence, as is well known (see [7]), there is an invertible bicontinuous selfadjoint linear operator $Q \in B(F)$ such that $[y, z] = (Qy, Qz) \forall y, z \in F$. Now $\forall \mu \in S_A,$

$$
\|Q\hat{T}_\mu^{-1}(y)\|^2 = \|\hat{T}_\mu^{-1}(y)\|^2 = \langle m(\nu), \|\hat{T}_\mu^{-1}Q^{-1}(y)\|^2 \rangle = \langle m(\nu), \|\hat{T}_\nu^{-1}Q^{-1}(y)\|^2 \rangle = \|Q^{-1}(y)\|^2 = \|y\|^2
$$

by left invariance of $m$. Therefore $U_\mu = Q\hat{T}_\mu^{-1}$ is unitary for any $\mu \in S_A$. This implies that \{\hat{T}_\mu : \mu \in S_A\} is similar to a unitary representation on $F$. Finally, since $\mu \rightarrow (\hat{T}_\mu, y, z)$ is bounded linear on $A$, its restriction to $S_A$ is in $AUC(S_A)$. To show that the same is true for the inverse representation, we first observe that since $F$ is finite dimensional, the strong operator topology and the uniform operator topology agree on $B(F)$. Hence $\mu \rightarrow \hat{T}_\mu^{-1}$ is also continuous when $B(F)$ has the uniform operator topology. Consequently given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\mu - \nu\| < \delta$ implies $\|\hat{T}_\mu^{-1} - \hat{T}_\nu^{-1}\| < \varepsilon$. In particular, if $\mu, \nu \in S_A$ and $\|\mu - \nu\| < \delta$, then $\|U_\mu - U_\nu\| = \|Q\hat{T}_\mu^{-1}Q^{-1} - Q\hat{T}_\nu^{-1}Q^{-1}\| \leq \|Q\| \cdot \|Q^{-1}\| \cdot \varepsilon$. But $\hat{T}_\mu^{-1} = Q\hat{T}_\mu^{-1}Q$ $\forall \mu \in S_A$, since $U_\mu$ is unitary for such $\mu$ and $Q$ is self-adjoint. Therefore

$$
\|\hat{T}_\mu^{-1} - \hat{T}_\nu^{-1}\| \leq \|Q\|^2 \cdot \|Q^{-1}\|^2 \cdot \varepsilon \ \forall \mu, \nu \in S_A
$$

with $\|\mu - \nu\| < \delta$. This implies that the function $\mu \rightarrow y^*\hat{T}_\mu^{-1}y$ on $S_A$ is in $AUC(S_A)$ $\forall y \in F, \ y^* \in F^*.$

**Theorem 3.4.** Let $A$ be an $F$-algebra.

(a) If $A$ is left amenable, then $A$ satisfies property $T(n)$ for $n = 1, 2, \ldots$ where property $T(n)$ is defined as follows: Let $E$ be a separated locally convex space and $T : A \times E \rightarrow E$ be a representation of $A$ as linear operators in $E$. Let $X$ be a subset of $E$ such that there exists a closed $S_A$-invariant subspace $H$ of $E$ of codimension $n$ and $(x + H) \cap X$ is compact convex for each $x \in E$. If the action of $S_A$ on $E$ is inversely equicontinuous modulo $H$ and $\mathcal{L}_n(X)$ is nonempty and $S_A$-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_{\mu}(L_0) = L_0$ $\forall \mu \in S_A$.

(b) If $S$ satisfies property $T(1)$, then $A$ is left amenable (hence $A$ satisfies $T(n)$ for every $n$).

**Proof.** (a) With the notation of Lemma 3.1, define the affine representation $\Psi : S_1 \times K \rightarrow K$ by $\Psi_\mu(B) = T_\mu \circ B \circ \hat{T}_\mu^{-1}, \mu \in S_A$ and $B \in K$. We want to show that $\psi$ is an $A$-representation for the pair $S_A$ and $AUC(S_A)$ in the sense of Argabright [1, §2]. That is, for each $h \in A(K)$, the affine continuous functions on $K$, the function $\mu \rightarrow h(\Psi_\mu(B))$ is in $AUC(S_A)$ for each $B \in K$. By Argabright [1, Lemma 1] and Kelley and Namioka [9, Theorem 14.6, p. 120], it is easy to show that $h$ is in $A(K)$ of the form $h(B) = x^* B y$, $x \in E^*$ and $y \in F$. Let $\{e_1, \ldots, e_n\}$ be a basis in $F$. Then $\{qT_\mu B e_1, \ldots, qT_\mu B e_n\}$ is also a basis in $F$ for any $\mu \in S_A$ and $B \in K$ (by definition of $K$, $S_A$-invariance of $\mathcal{L}_n(X)$ and the fact that $(x + H) \cap X$ is compact and convex for each $x \in E$). Write $y = \sum_{j=1}^n \alpha_j(\mu) qT_\mu B e_j$ where $\mu \in S_A, B \in K$. Note that the scalars $\alpha_j(\mu)$ depend only on $\mu \in S_A$ and not on $B$. (See Fan [4, equation (7), p. 449].) Then $\hat{T}_\mu^{-1}(y) = \sum_{j=1}^n \alpha_j(\mu) e_j, \mu \in S_A$. (Since $q \circ B = \text{identity on } F$.) By Lemma 3.3, the functions $\mu \rightarrow \alpha_k(\mu) = y_k^*\hat{T}_\mu^{-1}(y)$ on
INVARIANT SUBSPACES AND AMENABLE LOCALLY COMPACT GROUPS

Sa belong to AUC(Sa) where \( \{y_1^*, \ldots, y_n^*\} \) is the basis dual to \( \{\tilde{e}_1, \ldots, \tilde{e}_n\} \). Now

\[
h(\Psi_\mu(B)) = x^*(T_\mu \circ B \circ \tilde{T}_\mu^{-1})y = \sum_{j=1}^{n} \alpha_j(\mu)x^*T_\mu B(\tilde{e}_j), \quad \mu \in S_A.
\]

Therefore the function \( \mu \rightarrow h(\Psi_\mu(B)) \) on \( S_A \) is in AUC(Sa). Hence \( \Psi : S_1 \times K \rightarrow K \) is an \( A \)-representation for the pair \( S_1, \text{AUC}(S_a) \). By Argabright [1, Theorem 1], \( \Psi \) has a common fixed point \( Q_0 \in K \). Put \( L_0 = Q_0(F) \in L_n(X) \), then \( T_\mu(L_0) = L_0 \) \( \forall \mu \in S_A \).

(b) Define \( E = A^{**} \) with the weak* topology and \( T : A \times E \rightarrow E \) by \( T_\mu(m) = \ell^*_\mu m \) where \( \ell^*_\mu \) is the adjoint of the map \( \ell_\mu : A^* \rightarrow A^* \) such that \( \ell_\mu(F)(\nu) = F(\mu \ast \nu), \nu \in A \). As in Lau [10], let \( X \) be the union of all one-dimensional subspaces of \( E = A^{**} \) generated by the means (states) on \( A^* \) and \( H = \{m \in A^{**} : m(I) = 0\} \). The arguments used in the proof of Lau [7, Theorem 1(b)] show that the hypotheses in property \( T(1) \) are all satisfied except the part on inverse equicontinuity modulo \( H \) of \( S_A \), which is also satisfied because the induced action \( \tilde{T}_\mu \) is independent of \( \mu \in S_A \). (\( \ell_\mu(I) = I \) if \( \mu \in S_A \), since \( I(\mu) = 1 \), if \( \mu \in S_A \).) Since \( S \) satisfies \( T(1) \), this means \( \ell^*_\mu(L_0) = L_0 \) \( \forall \mu \in S_A \), where \( L_0 \) is a one-dimensional subspace generated by some mean \( m_0 \). Necessarily, \( m_0 \) is a topological left invariant mean on \( A^* \).

A locally compact group \( G \) is amenable if the space \( CB(G) \) has a left invariant mean. Examples of amenable locally compact groups include all solvable groups, abelian groups and all compact groups. However, if \( G \) contains the free group on two generators as a closed subgroup, then \( G \) is not amenable (see Greenleaf [7] or Pier [13] for details).

THEOREM 3.5. Let \( G \) be a locally compact group. If \( G \) is amenable, then the group algebra \( L_1(G) \) and the measure algebra \( M(G) \) satisfy \( T(n) \) for each \( n = 1, 2, 3, \ldots \). Conversely, if either \( L_1(G) \) or \( M(G) \) satisfies \( T(1) \), then \( G \) is amenable.

PROOF. It follows from Greenleaf [7, Theorem 2.2.1] and Wong [17, Theorem 3.3] that amenability of \( G \) is equivalent to the left amenability of \( L_1(G) \) and \( M(G) \).

THEOREM 3.6. Let \( G \) be a locally compact group. Then both the Fourier algebra \( A(G) \) and the Fourier Stieltjes algebra \( B(G) \) have property \( T(n) \) for each \( n = 1, 2, 3, \ldots \).

PROOF. In this case, if \( A = A(G) \) or \( B(G) \), the \( A \) is a commutative \( F \)-algebra (see Eymard [3]). Hence \( S_A \) is commutative and so \( l_\infty(S_A) \), the space of bounded complex-valued functions on the discrete semigroup \( S_A \), has an invariant mean (see Day [2]). So by Lemma 2.1, \( A \) is left amenable. Now apply Theorem 3.4.

Given a semigroup \( S \), let \( l_1(S) \) be the Banach algebra as defined in Day [2] or Hewitt and Zuckermann [8]. \( S \) is called left amenable if \( l_\infty(S) \) has a left invariant mean (see Day [2]).

THEOREM 3.7. Let \( S \) be a semigroup. If \( S \) is left amenable, then the Banach algebra \( l_1(S) \) satisfies \( T(n) \) for each \( n = 1, 2, 3, \ldots \). Conversely if \( l_1(S) \) satisfies \( T(1) \), then \( S \) is left amenable.

PROOF. Any left invariant mean on \( l_\infty(S) \) is necessarily topological left invariant. Now apply Theorem 3.4 again.
REMARKS. (a) Left amenability of the Fourier algebra $A(G)$ of a locally compact group $G$ was proved by Renaud [14, Theorem 4] (see also Granirer [6, Proposition 5(a)]) by a completely different method.

(b) Theorem 3.7 can be generalized to all locally compact semigroups $S$ with left amenability of $S$ replaced by topological left amenability of the measure algebra $M(S)$ (see [16]), and $l_1(S)$ replaced by $M(S)$.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA T2N 1N4