THE STRUCTURE OF BOUNDED BILINEAR FORMS ON PRODUCTS OF C*-ALGEBRAS

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(Communicated by John B. Conway)

ABSTRACT. Let $A_1$ and $A_2$ be $C^*$-algebras and $B: A_1 \times A_2 \to \mathbb{C}$ be a bounded bilinear form. It is proved that there exist a Hilbert space $H$, two Jordan morphisms $\mu_i: A_i \to L(H)$, $i = 1, 2$, and two vectors $\xi_1, \xi_2 \in H$ such that

$$B(x, y) = (\mu_1(x)\xi_1 | \mu_2(y^*)\xi_2) \quad \text{for all} \ x \in A_1, \ y \in A_2.$$  

The proof depends on the Grothendieck-Pisier-Haagerup inequality and Halmos's unitary dilation theorem. An extremely elementary proof of the latter is given.

1. Introduction. In [8, Remark 5.3(a)] it was observed that some arguments in the proof of [8, Theorem 5.2] yield (essentially, cf. Remark 2.2) the following representation theorem for bounded bilinear forms on the Cartesian product of two $C^*$-algebras.

1.1. LEMMA. Let $A_1$ and $A_2$ be two $C^*$-algebras and $B: A_1 \times A_2 \to \mathbb{C}$ be a bounded bilinear form. Then there exist four Hilbert spaces $K'_i, K''_i$, $i = 1, 2$, cyclic *-representations $\pi'_i: A_i \to L(K'_i)$ with unit cyclic vectors $\xi'_1 \in K'_i$, cyclic *-antirepresentations $\pi''_i: A_i \to L(K''_i)$ with unit cyclic vectors $\xi''_1 \in K''_i$ and a bounded linear map $T: K'_1 \oplus K''_1 \to K'_2 \oplus K''_2$ with $\|T\| \leq \|B\|$ such that for all $x \in A_1, \ y \in A_2$

$$B(x, y) = (T(\pi'_1(x)\xi'_1, \pi''_2(y^*)\xi''_2) | (\pi'_2(y^*)\xi'_2, \pi''_1(x)\xi''_1))_{K'_1 \oplus K''_1}.$$  

Here and elsewhere, for any Hilbert space $H$, $(\cdot | \cdot)_H$ or $(\cdot | \cdot)$ denotes the inner product and $L(H)$ the space of bounded linear operators on $H$. If $K$ is a closed subspace of $H$, $P_K$ is the orthogonal projection of $H$ onto $K$.

The aim of this note is to point out that a representation in terms of Jordan morphisms can be obtained even without the help of the operator $T$. In the formulation of Theorem 2.1 the term "Jordan morphism" is only used for euphony; Jordan morphisms are known to be precisely the direct sums of *-representations and *-antirepresentations [12], and the proof actually yields such direct sums.

In the appendix we spell out for completeness the details of a proof of Lemma 1.1, since the proof is short and our point of view differs somewhat from that of [8]. We also find it worthwhile to communicate an utterly short and elementary proof of the Halmos dilation theorem used in the proof of Theorem 2.1.
2. The main result.

2.1. Theorem. Let $A_1$ and $A_2$ be $C^*$-algebras and $B: A_1 \times A_2 \to \mathbb{C}$ be a bounded bilinear form. There exist a Hilbert space $H$, two Jordan morphisms $\mu_i: A_i \to L(H)$, $i = 1, 2$, and two vectors $\xi_1, \xi_2 \in H$ such that

$$B(x, y) = (\mu_1(x)\xi_1|\mu_2(y^*)\xi_2) \quad \text{for all } x \in A_1, y \in A_2.$$

Proof. We use Lemma 1.1 and its notation, and write $K = K_1 \oplus K_2$ where $K_i = K'_i \oplus K''_i$, $i = 1, 2$. We modify a technique used in the proof of Theorem 2.4 in [4]. Define $\bar{T}: K \to K$ by the formula $\bar{T}(u, v) = (0, Tu)$. We may assume that $||B|| \leq 1$, so that $||\bar{T}|| \leq 1$. Thus $\bar{T}$ has a unitary dilation, i.e., there is a Hilbert space $H$ containing $K$ as a subspace such that $T = P_K U|K$ for some unitary operator $U: H \to H$ (see §3 for a proof and references). Write $H$ in the form $H = K_1 \oplus K_2 \oplus K_1^\perp$, denote $\pi_i = \pi_i' \oplus \pi_i'': A_i \to L(K_i)$ for $i = 1, 2$, and define $\mu_1(x) = (\pi_1(x)\xi_1, 0, 0)$ for $x \in A_1$, $\xi_1 \in K_1$, $\xi_2 \in K_2$, $\xi_3 \in K_1^\perp$, and $\mu_2(y) = U^*(0 \oplus \pi_2(y) \oplus 0)|U$ for $y \in A_2$. Moreover, denote $\xi_1 = ((\xi'_1, \xi''_1), 0, 0)$ and $\xi_2 = U^*(0, (\xi_2', \xi_2''), 0)$. A direct calculation using (1) in Lemma 1.1 now gives

$$(\mu_1(x)\xi_1|\mu_2(y^*)\xi_2)_H
= (U(\pi_1(x)\xi_1', 0, 0)|0, \pi_2(y^*)(\xi_2', 0, 0))_H
= (\bar{T}(\pi_1(x)\xi_1', 0)|0, \pi_2(y^*)(\xi_2', 0, 0))_K
= (\bar{T}(\pi_1(x)\xi_1', 0)|\bar{T}(\pi_2(y^*)(\xi_2', 0, 0)))_K = B(x, y). \quad \Box$$

2.2. Remark. Applying Theorem 2.1 to $\tilde{B}: A_2 \times A_1 \to \mathbb{C}$ defined by $\tilde{B}(y, x) = B(x, y)$ we see that $B$ can also be represented in the form $\tilde{B}(x, y) = (\mu_1(x)\mu_2(y)\xi_1|\xi_2)$, where $\mu_i: A_i \to L(H)$ are Jordan morphisms for some Hilbert space $H$, and $\xi_i \in H$ for $i = 1, 2$. Similarly, the technique of the above proof shows that in [3, Theorem 2.1, condition (3)], one may take $H = K$ and leave the operator $T$ out. Theorem 2.1 of the present paper was announced at the UCLA Functional Analysis seminar October 8, 1986; I am indebted to E. G. Effros for bringing a preprint of [3] in that connection to my attention.

3. Appendix: alternate proofs of essentially known results. The unitary dilation result in [7, Problem 177(a)] due to Halmos [6] is contained in part (b) of the following proposition. Unlike [6, 7] or the proof of (a) in [2], the proof below does not depend on the existence of the square root of a positive operator. The proof of (a) may actually be seen as flowing from a comparison of [2] with a more traditional approach (see e.g. [1, 9–11]) to the kind of problem treated there.

3.1. Proposition. Let $H_1$ and $H_2$ be Hilbert spaces and $T: H_1 \to H_2$ a linear contraction.

(a) There exist a Hilbert space $K_2$ containing $H_2$ as a subspace, and an isometric linear map $V: H_1 \to K_2$ such that $T = P_{K_2} V$.

(b) There exist Hilbert spaces $K_1$ and $K_2$ and an isometric linear surjection $U: K_1 \to K_2$ such that $K_i$ contains $H_i$ as a subspace, $i = 1, 2$, and $T = P_{K_2} U|H_1$.

Proof. (a) Denote $h(x, y) = (x|y) - (Tx|Ty)$ for $x, y \in H_1$. Then $h: H_1 \times H_1 \to \mathbb{C}$ is sesquilinear, and positive since $||T|| \leq 1$. Denote $N = \{x \in H_1|h(x, x) = 0\}$, and complete $H_1/N$ with the inner product $(x + N|y + N) = h(x, y)$ to a
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Hilbert space \( K \). Finally define \( K_2 = K \oplus H_2 \), and \( Vx = (x + N, Tx) \in K_2 \) for \( x \in H_1 \). Interpreting \( H_2 \) as a subspace of \( K_2 \), we have \( T = P_{H_2} V \), and \( \|Vx\|^2 = \|x + N\|^2 + \|Tx\|^2 = \|x\|^2 \).

(b) Continuing from (a), denote \( K_1 = H_1 \oplus (K_2 \ominus V(H_1)) \), and define \( U(x, y) = Vx + y \) for \( x \in H_1, y \in K_2 \ominus V(H_1) \). □

PROOF OF LEMMA 1.1. We may assume that \( \|B\| \leq 1 \). Denote \( B_0(x, y) = B(x, y^*) \), so that \( B_0 \) is a bounded sesquilinear form with \( \|B_0\| \leq 1 \). Using Haagerup’s general version [5, Theorem 1.1] of Pisier’s Grothendieck type inequality we get four states \( \varphi_i, \psi_i : A_i \to C, i = 1, 2 \), such that

\[
|B_0(x, y)| \leq \left[ \varphi_1(x^* x) + \psi_1(x x^*) \right]^{1/2} \left[ \varphi_2(y^* y) + \psi_2(y y^*) \right]^{1/2}
\]

for all \( x \in A_1, y \in A_2 \). (In a similar estimate for \( B \) exchange the roles of the last two states.) Denoting \( h_i(u, v) = \varphi_i(v^* u) + \psi_i(u v^*) \) and \( N_i = \{u \in A_i | h_i(u, u) = 0\} \), we get two inner product spaces \( A_i/N_i \) with the inner products \( \langle u + N_i, v + N_i \rangle = h_i(u, v) \). Completing these to Hilbert spaces \( K_i \) we obtain a well-defined bounded sesquilinear form \( \tilde{B}_0 : K_1 \times K_2 \to C \) with \( \|\tilde{B}_0\| \leq 1 \) and \( \tilde{B}_0(x + N_1, y + N_2) = B_0(x, y), x \in A_1, y \in A_2 \). Thus there is a bounded linear operator \( T : K_1 \to K_2 \) such that \( \tilde{B}_0(w, z) = (Tw|z), w \in K_1, z \in K_2 \). On the other hand, applying the GNS-construction [13, Theorem 9.14] to \( A_i \) and \( \varphi_i \), and to the opposite \( C^* \)-algebra (i.e., otherwise the same as \( A_i \) but equipped with the product \( u \cdot v = vu \) where \( vu \) is the product of \( A_i \)) and \( \psi_i \) we see that \( \varphi_i(u) = (\pi^i(u) \xi_i^* \xi_i') \) and \( \psi_i(u) = (\pi^{i'}(u) \xi_i' \xi_i'') \) for a cyclic *-representation \( \{\pi_i, K_i, \xi_i\} \) and a cyclic *-antirepresentation \( \{\pi^{i'}, K_i', \xi_i'\} \) of \( A_i \). A straightforward calculation shows that \( (u + N_i|v + N_i) \) is the same as the inner product of \( \pi_i(u) \xi_i \pi^{i'}(v) \xi_i'' \) and \( \pi^{i'}(v) \xi_i' \pi_i(u) \xi_i'' \) in \( K_i \oplus K_i'' \), which implies that we get for each \( i = 1, 2 \) a well-defined isometric linear map \( V_i : K_i \to K_i' \oplus K_i'' \) satisfying \( V_i(u + N_i) = (\pi_i(u) \xi_i, \pi^{i'}(u) \xi_i') \) for \( u \in A_i \). It is now easy to verify that the choice \( T = V_2 TV_1^* \) will work. □

REFERENCES


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