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SIMPLE PROOFS OF BERNSTEIN-TYPE INEQUALITIES

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ABSTRACT. A polynomial identity is established by the use of Lagrange interpolation. This identity is used to obtain simple proofs of Bernstein-type inequalities, one of which is an improvement of a recent result of Frappier, Rahman, and Ruscheweyh.

Let $\mathcal{P}_n$ denote the collection of complex algebraic polynomials of degree at most n. For $P \in \mathcal{P}_n$ we will write

$$\|P\| = \max_{|z|=1} |P(z)|.$$

A well-known result of S. Bernstein (for references see [**11**]) states that if $P \in \mathcal{P}_n$ then $\|P'\| \leq n\|P\|$.

The following theorems are three of the many improvements of Bernstein's inequality that are known.

THEOREM A. *If $P \in \mathcal{P}_n$ and $\widetilde{P}(z) = z^n\overline{P(1/\bar{z})}$, then*

$$\max_{|z|=1}[|\widetilde{P}'(z)| + |P'(z)|] = n\|P\|.$$

THEOREM B. *If $P \in \mathcal{P}_n$ and $z_1, \ldots, z_{2n}$ are any $2n$ equally spaced points on the unit circle, then $\|P'\| \leq n \max_{1\leq k\leq 2n} |P(z_k)|$.*

THEOREM C. *If $P \in \mathcal{P}_n$, then $\|P'\| \leq n\|\operatorname{Re}(P)\|$.*

In establishing the equality in Theorem A, the nontrivial part is to prove that the left side is less than or equal to the right side. This inequality appears in [**4** and **8**] and generalizations of this inequality can be found in [**1** and **2**]. As noted in [**5**] an immediate corollary of Theorem A is the fact that if P is self-inversive (i.e., $\widetilde{P} = uP$ for some u with $|u| = 1$) then $\|P'\| = (n/2)\|P\|$ (also proved in [**3**, **7** and **10**]). Theorem B appears in [**1**] where the authors also show that $2n$ is the smallest number of equally spaced points that can be used. Theorem C is due to G. Szegö [**12**] and later proofs can be found in [**1**, **4** and **9**].

The purpose of this paper is to show that Theorems A, B and C are immediate corollaries of an interesting identity, which itself is elementary to prove. Moreover this identity will yield the following improved version of Theorem B.

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THEOREM B′. *Let $z_1, \ldots, z_{2n}$ be any $2n$ equally spaced points on the unit circle, say $z_k = ue^{k\pi i/n}$, $|u| = 1$, $1 \le k \le 2n$. If $P \in \mathcal{P}_n$, then*

$$\|P'\| \le \frac{n}{2}\left[\max_{k \text{ odd}} |P(z_k)| + \max_{k \text{ even}} |P(z_k)|\right].$$

As an example of a case comparing Theorems B and B′, consider $P(z) = z^n - 1$. With z_k as described above it is clear that $P(z_k) = u^n - 1$ for k even and $P(z_k) = -(u^n + 1)$ for k odd. Therefore

$$\max_{1\le k\le 2n} |P(z_k)| = \max\{|u^n - 1|, |u^n + 1|\} := f(u),$$

and

$$\tfrac{1}{2}\left[\max_{k \text{ odd}} |P(z_k)| + \max_{k \text{ even}} |P(z_k)|\right] = \tfrac{1}{2}[|u^n + 1| + |u^n - 1|] := g(u).$$

It is easy to show that $g(u) \le \sqrt{2} \le f(u)$. Moreover when $u^n = \pm 1$ then $g(u) = 1$ and hence for this special P the inequality in Theorem B′ is sharp.

The identity which is central to this paper was motivated by the methods in [**6**]. It is stated in the following theorem.

THEOREM. *Suppose λ is any complex number with $|\lambda| = 1$ and let $t_1, \ldots, t_n$ be the nth roots of λ. If $P \in \mathcal{P}_n$ then for all z with $|z| = 1$,*

$$nP(z) - zP'(z) + \frac{\lambda}{z^{n-1}}P'(z) = \frac{1}{n}\sum_{k=1}^{n} P(t_k)\left|\frac{z^n - \lambda}{z - t_k}\right|^2 \tag{1}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left|\frac{z^n - \lambda}{z - t_k}\right|^2 = n. \tag{2}$$

PROOF. Suppose $t_1, \ldots, t_n$ are any n distinct complex numbers and let $\omega(z) = (z - t_1)\cdots(z - t_n)$. Note that if $P(z) = a_n z^n + \cdots + a_0$, then, regardless of whether or not $a_n = 0$, the polynomial defined by $P(z) - a_n\omega(z)$ is in $\mathcal{P}_{n-1}$. Therefore by using the Lagrange interpolation formula we can write

$$P(z) - a_n\omega(z) = \sum_{k=1}^{n} P(t_k)\frac{\omega(z)}{\omega'(t_k)(z - t_k)}.$$

If we divide this last identity by $\omega(z)$ and then differentiate we find that

$$P(z)\omega'(z) - P'(z)\omega(z) = \sum_{k=1}^{n} P(t_k)\frac{\omega^2(z)}{\omega'(t_k)(z - t_k)^2}. \tag{3}$$

In particular when $t_1, \ldots, t_n$ are the nth roots of λ, so that $\omega(z) = z^n - \lambda$, then (3) becomes

$$nz^{n-1}P(z) - (z^n - \lambda)P'(z) = \frac{1}{\lambda n}\sum_{k=1}^{n} P(t_k)\left(\frac{z^n - \lambda}{z - t_k}\right)^2 t_k. \tag{4}$$

Note that if $|u| = |v| = 1$ then $(u - v)^2 = -|u - v|^2 uv$. Therefore if $|z| = |\lambda| = 1$ then

$$\left(\frac{z^n - \lambda}{z - t_k}\right)^2 t_k = \left|\frac{z^n - \lambda}{z - t_k}\right|^2 \lambda z^{n-1},$$

and (4) can be rewritten to obtain (1). To derive (2) use (1) with $P(z) = z^n$.

COROLLARY. *Suppose λ is any complex number with $|\lambda| = 1$. Let $t_1, \ldots, t_n$ be the nth roots of λ and $s_1, \ldots, s_n$ be the nth roots of $-\lambda$. If $P \in \mathcal{P}_n$, then for all z with $|z| = 1$*

$$\frac{2\lambda}{z^{n-1}} P'(z) = \frac{1}{n} \sum_{k=1}^{n} P(t_k) \left| \frac{z^n - \lambda}{z - t_k} \right|^2 - \frac{1}{n} \sum_{k=1}^{n} P(s_k) \left| \frac{z^n + \lambda}{z - s_k} \right|^2. \tag{5}$$

PROOF. Replace λ by $-\lambda$ in (1) and subtract the resulting identity from (1).

We can now prove Theorems A, B′ and C.

PROOF OF THEOREM A. It follows from (1) and (2) that for all λ, z with $|\lambda| = |z| = 1$,

$$|[nP(z) - zP'(z)] + (\lambda/z^{n-1})P'(z)| \leq n\|P\|.$$

For each z, $|z| = 1$, we can choose a corresponding λ, $|\lambda| = 1$, so that the left side of this inequality becomes $|nP(z) - zP'(z)| + |P'(z)|$. Thus it follows that

$$\max_{|z|=1} [|nP(z) - zP'(z)| + |P'(z)|] \leq n\|P\|.$$

The reverse inequality follows easily since for $|z| = 1$,

$$|nP(z)| = |nP(z) - zP'(z) + zP'(z)| \leq |nP(z) - zP'(z)| + |P'(z)|.$$

Therefore we have that

$$\max_{|z|=1} [|nP(z) - zP'(z)| + |P'(z)|] = n\|P\|.$$

This establishes Theorem A since it is easy to show that $|\widetilde{P}'(z)| = |nP(z) - zP'(z)|$ for $|z| = 1$.

PROOF OF THEOREM B′. With $z_k = ue^{k\pi i/n}$, $|u| = 1$, $1 \leq k \leq 2n$, let $t_k = z_{2k}$ and $s_k = z_{2k-1}$, $1 \leq k \leq n$. Note that $t_k^n = u^n$ and $s_k^n = -u^n$, $1 \leq k \leq n$. Theorem B′ is obtained easily by using (5) with $\lambda = u^n$ and applying the triangle inequality along with (2).

PROOF OF THEOREM C. Equate real parts in (5) and use the triangle inequality along with (2) to obtain

$$2|\operatorname{Re}[(\lambda/z^{n-1})P'(z)]| \leq n\|\operatorname{Re}(P)\| + n\|\operatorname{Re}(P)\|, \qquad |\lambda| = |z| = 1.$$

Choose z_0 so that $|P'(z_0)| = \|P'\|$ and write $P'(z_0) = \varepsilon\|P'\|$, $|\varepsilon| = 1$. Theorem C now follows from the last inequality by putting $z = z_0$ and $\lambda = z_0^{n-1}/\varepsilon$.

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