CONVERGENCE AND INTEGRABILITY OF DOUBLE TRIGONOMETRIC SERIES WITH COEFFICIENTS OF BOUNDED VARIATION

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ABSTRACT. We prove that if \( c(j, k) \to 0 \) as \( \max(|j|, |k|) \to \infty \) and

\[
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j, k)| < \infty,
\]

then the series \( \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j, k) e^{i(jx + ky)} \) converges both pointwise for every \((x, y) \in (T \setminus \{0\})^2\) and in the \( L^p(T^2)\)-metric for \( 0 < p < 1 \), where \( T \) is the one-dimensional torus. Both convergence statements remain valid for the three conjugate series under these same coefficient conditions.

1. Introduction. We will consider double trigonometric series of the form

\[
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j, k) e^{i(jx + ky)}
\]

where \( \{c(j, k) : -\infty < j, k < \infty\} \) is a null sequence of complex numbers. Here and in the sequel,

\[
(x, y) \in T^2 := \{(x, y) \in R^2 : 0 \leq x, y < 2\pi\}
\]

the two-dimensional torus, whereas \( T := \{x \in R : 0 \leq x < 2\pi\} \).

The pointwise convergence of series (1.1) is usually defined in Pringsheim's sense (see, e.g. [5, Vol. 2, Chapter 17]). This means that we form the symmetric partial sums

\[
s_{MN}(x, y) := \sum_{j=-M}^{M} \sum_{k=-N}^{N} c(j, k) e^{i(jx + ky)} \quad (M, N \geq 0)
\]

and then let both \( M \) and \( N \) tend to \( \infty \), independently of one another, and assign the limit \( f(x, y) \) (if it exists) to series (1.1) as its sum.

Following Hardy [1], series (1.1) is said to be regularly convergent if

(i) it converges in Pringsheim's sense, and

(ii) the single series

\[
\sum_{j=-\infty}^{\infty} c(j, k) e^{i(jx + ky)}
\]

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converges for each fixed value of \( k \), and the single series
\[
\sum_{k=\infty}^{\infty} c(j, k)e^{i(jx+ky)}
\]
converges for each fixed value of \( j \).

As is known, if (i) and (ii) are satisfied, then the sum \( f(x, y) \) of series (1.1) can be computed in the way of iterated summations, too:

\[
f(x, y) = \sum_{j=\infty}^{\infty} \left[ \sum_{k=\infty}^{\infty} c(j, k)e^{i(jx+ky)} \right]
= \sum_{k=\infty}^{\infty} \left[ \sum_{j=\infty}^{\infty} c(j, k)e^{i(jx+ky)} \right] .
\]

The notion of regular convergence was rediscovered in [2] where it was defined by the following equivalent condition (and called "convergence in a restricted sense"): the sums
\[
\sum_{M_1 \leq |j| \leq M_2} \sum_{N_1 \leq |k| \leq N_2} c(j, k)e^{i(jx+ky)} \to 0
\]
as \( \max(M_1, N_1) \to \infty \), independently of the choices of \( M_2 \) and \( N_2 \) where \( 0 \leq M_1 \leq M_2 \) and \( 0 \leq N_1 \leq N_2 \).

Now we introduce an even stronger notion of convergence, we may call it strongly regular convergence, which requires that the sums (with \( M_1 \leq M_2 \) and \( N_1 \leq N_2 \))

\[
s(Q; x, y) := \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c(j, k)e^{i(jx+ky)} \to 0
\]
in each of the following limiting cases:

\[
\begin{cases}
(i) & M_1 \to \infty, \text{ while } M_2, N_1, \text{ and } N_2 \text{ are arbitrary.} \\
(ii) & M_2 \to -\infty, \text{ while } M_1, N_1, \text{ and } N_2 \text{ are arbitrary.} \\
(iii) & N_1 \to \infty, \text{ while } N_2, M_1, \text{ and } M_2 \text{ are arbitrary.} \\
(iv) & N_2 \to -\infty, \text{ while } N_1, M_1, \text{ and } M_2 \text{ are arbitrary.}
\end{cases}
\]

Here and in the sequel, \( Q \) denotes the set of the lattice points of the plane contained in the rectangle \( \{(j, k): M_1 \leq j \leq M_2 \text{ and } N_1 \leq k \leq N_2 \} \).

After these convergence notions, we repeat the definitions of the three conjugate series to (1.1):

\[
\sum_{j=\infty}^{\infty} \sum_{k=\infty}^{\infty} (-i \text{ sgn } j)c(j, k)e^{i(jx+ky)}
\]
(conjugate with respect to \( x \)),

\[
\sum_{j=\infty}^{\infty} \sum_{k=\infty}^{\infty} (i \text{ sgn } k)c(j, k)e^{i(jx+ky)}
\]
(conjugate with respect to \( y \)),
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(1.6) \[ \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i \text{sgn } j)(-i \text{sgn } k)c(j,k) e^{i(jx+ky)} \]

(conjugate with respect to \(x\) and \(y\)).

If series (1.4)–(1.6) converge in Pringsheim’s sense, then their sums are denoted by \(\tilde{f}^{(1,0)}(x,y)\), \(\tilde{f}^{(0,1)}(x,y)\), and \(\tilde{f}^{(1,1)}(x,y)\), respectively, and are called the corresponding conjugate functions to \(f(x,y)\) (see, e.g., [3]).

2. Main results. Let \(\{c(j,k)\} : -\infty < j, k < \infty\) be a double sequence. We remind the reader that its differences are defined by

\[ \Delta_{10}c(j,k) = c(j,k) - c(j+1,k), \]
\[ \Delta_{01}c(j,k) = c(j,k) - c(j,k+1), \]
\[ \Delta_{11}c(j,k) = c(j,k) - c(j+1,k) - c(j,k+1) + c(j+1,k+1), \]

and \(\{c(j,k)\}\) is said to be of bounded variation if

(2.1) \[ C_{11} := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11}c(j,k)| < \infty. \]

We will prove the following convergence statements.

THEOREM. Let \(\{c(j,k)\} : -\infty < j, k < \infty\) be a double sequence of complex numbers that is of bounded variation and such that

(2.2) \[ c(j,k) \to 0 \text{ as } \max(|j|, |k|) \to \infty. \]

Then series (1.1)

(i) converges pointwise in the strongly regular sense to some function \(f(x,y)\) for every \((x,y) \in (T\setminus\{0\})^2\);

(ii) converges in the \(L^p(T^2)\)-metric to \(f\) for every \(0 < p < 1\):

(2.3) \[ \left\| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c(j,k) e^{i(jx+ky)} - f(x,y) \right\|_p \to 0 \]

as \(M_2, N_2 \to \infty\) and \(M_1, N_1 \to -\infty\). In particular, we have \(f \in L^p(T^2)\) for every \(0 < p < 1\).

(iii) Analogous conclusions can be drawn for the conjugate series (1.4)–(1.6) and the corresponding conjugate functions \(\tilde{f}^{(1,0)}\), \(\tilde{f}^{(0,1)}\), \(\tilde{f}^{(1,1)}\), respectively.

Here and in the sequel, \(\| \cdot \|_p\) denotes the \(L^p(T^2)\)-norm defined by

\[ \|g\|_p := \left( \int_0^{2\pi} \int_0^{2\pi} |g(x,y)|^p dx dy \right)^{1/p}. \]

Our Theorem can be considered the extension of a theorem of Uljanov [4] from single to double trigonometric series.
3. Auxiliary results. We present two lemmas.

**Lemma 1.** If \( \{c(j,k)\} \) satisfies conditions (2.1) and (2.2), then for every \( k \),

\[
\sum_{j=-\infty}^{\infty} |\Delta_{10} c(j,k)| \leq C_{11} (< \infty),
\]

\[
\sum_{j=-\infty}^{\infty} |\Delta_{10} c(j,k)| \to 0 \quad \text{as } |k| \to \infty,
\]

\[
\sup_{|j|>M} \sum_{k} |\Delta_{10} c(j,k)| \to 0 \quad \text{as } M \to \infty.
\]

We note that analogous statements are available for the series \( \sum_{k=-\infty}^{\infty} |\Delta_{01} c(j,k)| \) under the same conditions.

**Proof.** By (2.2),

\[
\Delta_{10} c(j,k_0) = \sum_{k=k_0}^{\infty} \Delta_{11} c(j,k),
\]

whence

\[
\sum_{j=-\infty}^{\infty} |\Delta_{10} c(j,k_0)| \leq \sum_{j=-\infty}^{\infty} \sum_{k=k_0}^{\infty} |\Delta_{11} c(j,k)|
\]

and (3.1) follows.

Again by (2.2),

\[
\Delta_{10} c(j,k_0) = -\sum_{k=-\infty}^{k_0-1} \Delta_{11} c(j,k),
\]

whence

\[
\sum_{j=-\infty}^{\infty} |\Delta_{10} c(j,k_0)| \leq \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{k_0-1} |\Delta_{11} c(j,k)|.
\]

Clearly (3.2) follows from (2.1), (3.4), and (3.5).

Finally, (3.3) is a consequence of (3.2) (applied for large values of \( |k| \)) and (3.1) (applied for small values of \( |k| \)).

**Lemma 2.** If \( \{c(j,k)\} \) satisfies conditions (2.1) and (2.2), then the sequences

\[
\begin{align*}
(i) & \quad \{(-i \text{sgn } j)c(j,k)\}, \\
(ii) & \quad \{(-i \text{sgn } k)c(j,k)\}, \\
(iii) & \quad \{(-i \text{sgn } j)(-i \text{sgn } k)c(j,k)\}
\end{align*}
\]

also satisfy these same conditions.
PROOF. It suffices to prove (i), since (ii) is a symmetric counterpart of (i), while (iii) follows from a repeated application of (i) and (ii). In the case of (i), the fulfillment of (2.2) is obvious. Inequality (2.1) follows from Lemma 1:

\[
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11}((-i \text{ sgn } j)c(j, k))| = \left\{ \sum_{j=-\infty}^{-2} \sum_{k=-\infty}^{\infty} + \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \right\} |\Delta_{11}c(j, k)| + \sum_{k=-\infty}^{\infty} |\Delta_{01}c(-1, k)| + \sum_{k=-\infty}^{\infty} |\Delta_{01}c(1, k)|.
\]

4. Proof of the Theorem. We will use the notation \( w(x) = 1 - e^{-ix} \). Then

\[
|w(x)| = 2 \sin \frac{x}{2} \quad \text{for } 0 \leq x < 2\pi
\]

and

\[
w(x)w(y) = 1 - e^{-ix} - e^{-iy} + e^{i(x+y)}.
\]

Part 1: Pointwise convergence. Performing an “Abel transformation-like” rearrangement yields, for any \( M_1 \leq M_2 \) and \( N_1 \leq N_2 \),

\[
w(x)w(y) \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c(j, k)e^{i(jx+ky)} = \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_1-1}^{N_2-1} \Delta_{11}c(j, k)e^{i(jx+ky)} + \sum_{j=M_1-1}^{M_2-1} \Delta_{10}c(j, N_2)e^{i(jx+N_2y)}
- \sum_{j=M_1-1}^{M_2-1} \Delta_{10}c(j, N_1-1)e^{i(jx+(N_1-1)y)} + \sum_{k=N_1-1}^{N_2-1} \Delta_{01}c(M_2, k)e^{i(M_2x+ky)}
- \sum_{k=N_1-1}^{N_2-1} \Delta_{01}c(M_1-1, k)e^{i((M_1-1)x+ky)} + c(M_2, N_2)e^{i(M_2x+N_2y)}.
\]
Hence, using (4.1) and notation (1.2),

$$|q(Q;x,y)| \leq \frac{1}{4\sin \frac{\pi}{2} \sin \frac{\pi}{2}} \left\{ \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_1-1}^{N_2-1} |\Delta_{11} c(j,k)| 
+ \sum_{j=M_1-1}^{M_2-1} \left[ |\Delta_{10} c(j,N_2)| + |\Delta_{10} c(j,N_1-1)| \right] 
+ \sum_{k=N_1-1}^{N_2-1} \left[ |\Delta_{01} c(M_2,k)| + |\Delta_{01} c(M_1-1,k)| \right] + \left| c(M_2,N_2) \right| \right\}.$$ 

Making use of Lemma 1, we can see that each term in the braces on the right-hand side tends to zero as any one of the limiting cases (i)–(iv) in (1.3) is realized. Thus, the sum $f(x,y)$ of series (1.1) certainly exists for all $0 < x, y < 2\pi$.

**Part 2: $L^p(T^2)$-convergence.** It is plain that

$$|f(x,y) - q(Q;x,y)| \leq \sum_{j=-\infty}^{M_1-1} \sum_{k=-\infty}^{\infty} c(j,k) e^{i(jx+ky)}$$

$$\leq \sum_{j=-\infty}^{M_2} \sum_{k=-\infty}^{\infty} c(j,k) e^{i(jx+ky)}$$

$$\leq \sum_{j=M_1}^{M_2} \sum_{k=-\infty}^{N_2-1} c(j,k) e^{i(jx+ky)}$$

$$\leq \sum_{j=M_2}^{\infty} \sum_{k=N_2+1}^{\infty} c(j,k) e^{i(jx+ky)}$$

Similarly to (4.2), for any $M > M_2$ and $N > 0$,

$$w(x)w(y) \sum_{j=M_2+1}^{M} \sum_{k=-N}^{N} c(j,k) e^{i(jx+ky)}$$

$$= \sum_{j=M_2}^{M-1} \sum_{k=-N-1}^{N-1} \Delta_{11} c(j,k) e^{i(jx+ky)} + \sum_{j=M_2}^{M-1} \Delta_{10} c(j,N) e^{i(jx+Ny)}$$

$$- \sum_{j=M_2}^{M-1} \Delta_{10} c(j,-N-1) e^{i(jx-(N+1)y)}$$

$$+ \sum_{k=-N-1}^{N} \Delta_{01} c(M,k) e^{i(Mx+ky)}$$

$$- \sum_{k=-N-1}^{N} \Delta_{01} c(M_2,k) e^{i(M_2x+ky)} + c(M,N) e^{i(Mx+Ny)}.$$
Letting $M, N \to \infty$, by Lemma 1 we get that
\[
\sum_{j=M_2+1}^{\infty} \sum_{k=-\infty}^{\infty} c(j, k) e^{i(jx+ky)} = \sum_{j=M_2}^{\infty} \sum_{k=-\infty}^{\infty} \Delta_{11} c(j, k) e^{i(jx+ky)} - \sum_{k=-\infty}^{\infty} \Delta_{01} c(M_2, k) e^{i(M_2 x+ky)}.
\]
Hence
\[
\sum_2 \leq \frac{1}{|w(x)w(y)|} \left\{ \sum_{j=M_2+1}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j, k)| + \sum_{k=-\infty}^{\infty} |\Delta_{01} c(M_2, k)| \right\}.
\]
By (4.1), for $0 < p < 1$,
\[
\left\| \frac{1}{w(x)w(y)} \right\|_p = \left[ \int_0^{2\pi} \int_0^{2\pi} \frac{dz \, dy}{4p \sin^p \frac{z}{2} \sin^p \frac{y}{2}} \right]^{1/p} < \infty.
\]
So, by (2.1) and the counterpart of (3.2) (when the roles of $j$ and $k$ are interchanged), we obtain that
\[
\left\| \sum_2 \right\|_p \to 0 \quad \text{as } M_2 \to \infty.
\]
In a similar way, we can deduce that
\[
\left\| \sum_1 \right\|_p \to 0 \quad \text{as } M_1 \to -\infty.
\]
Estimating $\sum_4$, let $N > N_2$. Then, following the pattern occurring in (4.2),
\[
\sum_{j=M_1}^{M_2} \sum_{k=N_2+1}^{N} c(j, k) e^{i(jx+ky)}
\]
\[
= \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{N-1} \Delta_{11} c(j, k) e^{i(jx+ky)}
\]
\[
+ \sum_{j=M_1-1}^{M_2-1} \Delta_{10} c(j, N) e^{i(jx+Ny)}
\]
\[
- \sum_{j=M_1-1}^{M_2-1} \Delta_{10} c(j, N_2) e^{i(jx+N_2y)}
\]
\[
+ \sum_{k=N_2}^{N-1} \Delta_{01} c(M_2, k) e^{i(M_2 x+ky)}
\]
\[
- \sum_{k=N_2}^{N-1} \Delta_{01} c(M_1-1, k) e^{i((M_1-1)x+ky)} + c(M_2, N) e^{i(M_2 x+Ny)}.
\]
Letting $N \to \infty$, by Lemma 1,
\[
\sum_{j=M_1}^{M_2} \sum_{k=N_2+1}^{\infty} c(j,k)e^{i(jx+ky)} = \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{\infty} \Delta_{11} c(j,k)e^{i(jx+ky)} \]
\[
- \sum_{j=M_1-1}^{M_2-1} \Delta_{10} c(j,N_2)e^{i(jx+N_2y)} \]
\[
+ \sum_{k=N_2}^{\infty} \Delta_{01} c(M_2,k)e^{i(M_2x+ky)} \]
\[
- \sum_{k=N_2}^{\infty} \Delta_{01} c(M_1-1,k)e^{i((M_1-1)x+ky)}. \]

Hence
\[
\sum_{4} \leq \frac{1}{|w(x)w(y)|} \left\{ \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{\infty} |\Delta_{11} c(j,k)| + \sum_{j=M_1-1}^{M_2-1} |\Delta_{10} c(j,N_2)| \right. \]
\[
+ \sum_{k=N_2}^{\infty} |\Delta_{01} c(M_2,k)| + \sum_{k=N_2}^{\infty} |\Delta_{01} c(M_1-1,k)| \left\}. \right.
\]

By (2.1), (3.2), and its counterpart, we get that
\[
\left\| \sum_{4} \right\|_p \to 0 \quad \text{as } N_2 \to \infty. \tag{4.9} \]

A similar argument gives that
\[
\left\| \sum_{3} \right\|_p \to 0 \quad \text{as } N_1 \to -\infty. \tag{4.10} \]

Combining (4.3), (4.6), (4.7), (4.9), and (4.10) yields (2.3).

Part 3: Convergence of the conjugate series. This immediately follows from (i) and (ii) of the Theorem, via Lemma 2.

REFERENCES


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