A NOTE ON PSEUDOCONVEXITY
AND PROPER HOLOMORPHIC MAPPINGS
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ABSTRACT. In this paper we discuss some connections between proper holo-
morphic mappings between domains in $\mathbb{C}^n$ and the boundary behaviors of
certain canonical invariant metrics. A compactness theorem has been proved.
This generalizes slightly an earlier result proved by the second author.

Introduction. A continuous mapping $f : X_1 \to X_2$ between two topological
spaces is called proper if $f^{-1}(K) \subseteq X_1$ is compact whenever $K \subseteq X_2$ is compact.
Proper holomorphic mappings between analytic spaces stand out for their beauty
and simplicity. For instance, if $g : D_1 \to D_2$ is a proper holomorphic mapping
between two bounded domains in $\mathbb{C}^n$, a theorem of Remmert says that $(D_1, g, D_2)$
is a finite branching cover. The branching locus in $D_1$ is described by $\{z \in D_1 \mid\det(dg(z)) = 0\}$. For the past ten years, there has been a great amount of activity in
characterizing the proper holomorphic mappings between pseudoconvex domains.
It has been known for a long time that there are numerous proper holomorphic maps
between unit disks in $\mathbb{C}^1$. The simplest example is $g : \Delta = \{z \in \mathbb{C}^1 \mid |z| < 1\} \to \Delta$, $g(z) = z^n$, where $n$ is any positive integer. Nevertheless, such a phenomenon is
no longer true in higher-dimensional cases. H. Alexander was able to verify the
following interesting fact.

THEOREM 1 [1]. Let $B_n = \{ (z_1, z_2, \ldots, z_n) \mid \sum_{i=1}^{n} |z_i|^2 < 1 \}$ be the unit ball in
$\mathbb{C}^n$, $n \geq 2$. Suppose $f : B_n \to B_n$ is a proper holomorphic mapping. Then $f$ must
be a biholomorphism.

The following result due to S. Pinčuk is an extension of Alexander's theorem.

THEOREM 2 [5]. Let $D_1$ and $D_2$ be two strongly pseudoconvex bounded domains
with smooth boundaries in $\mathbb{C}^n$, $n \geq 2$. Suppose $f : D_1 \to D_2$ is a proper holomorphic
mapping. Then $f$ is a covering.

In [7] the second author proved the following result concerning biholomorphic
groups of strongly pseudoconvex domains.

THEOREM 3 [7]. Let $D$ be a strongly pseudoconvex bounded domain with smooth
boundary in $\mathbb{C}^n$. Then $\text{Aut}(D)$ is noncompact iff $D$ is biholomorphic to $B_n$, $n = \dim \mathbb{C} D$.
In view of a lot of recent attention on the topic of proper holomorphic mappings, the authors feel that it might be worthwhile to point out the following startling fact which generalizes Theorem 3.

**Theorem 4.** Let $D_1$ and $D_2$ be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbb{C}^n$, $n \geq 2$. Then $P(D_1, D_2)$ is noncompact iff both $D_1$ and $D_2$ are biholomorphic to $B_n$, where $P(D_1, D_2)$ denotes the set of all proper holomorphic mappings between $D_1$ and $D_2$.

Pinčuk’s Theorem 2 says that proper holomorphic mappings between strongly pseudoconvex domains are unbranching. It follows that Theorem 4 is an immediate consequence of the local version stated next, which is the principal result of this note.

**Theorem 5.** Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$. We denote $P_0(D_1, D_2)$ as the set of all unbranching proper holomorphic maps from $D_1$ to $D_2$. Suppose the following two conditions are fulfilled.

1. There is a strongly pseudoconvex boundary point $p \in \partial D_2$.
2. There exists a point $x \in D_1$ and a sequence $\{f_j\} \subseteq P_0(D_1, D_2)$ such that $\{f_j(x)\}$ converges to $p$.

Then both $D_1$ and $D_2$ are biholomorphic to $B_n$.

**A** Some preliminaries and related results. Let $M$ be a complex manifold of dimension $n$, $x \in M$, and $k$ an integer between one and $n$.

**Definition.** The Eisenman differential $k$-measure on $M$ is a function $E_M^k: \bigwedge^k T(M) \to \mathbb{R}$ such that for all $(x, v) \in \bigwedge^k T_x(M)$,

$$E_M^k(x, v) = \inf \left\{ R^{-2k} \left| \text{there exists a holomorphic map } f: B_k(R) \to M \text{ such that } f(0) = x \text{ and } df_0 \left( \frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_2} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0) \right) = v \right\},$$

where $B_k(R) = \{w = (w_1, w_2, \ldots, w_k) \in C_k \mid \sum_{i=1}^{k} |w_i|^2 < R \}.$

When $k = 1$, it is called a Kobayashi-Royden differential metric $[6]$, denoted $K_M = k\sqrt{E_M^1}$. As $k = n$, it is a volume form, denoted by $E_M^n = |E_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|E_M^n|$ is a function on $M$.

On the other hand, the Carathéodory differential $k$-measure $C_M^k$ is defined as follows.

**Definition.** $C_M^k: \bigwedge^k T_x(M) \to \mathbb{R}$, $(x, v) \in \bigwedge^k T_x(M)$, $C_M^k(x, v) = \sup \{1/R^{2k} \mid \text{there exists a holomorphic mapping } f: M \to B_k(R) \text{ such that } f(x) = 0, df_0(x) = \partial/\partial w_1 \wedge \cdots \wedge \partial/\partial w_k(0) \}.$

When $k = 1$, it is called a Carathéodory-Reiffen differential metric, denoted by $C_M = \sqrt{C_M^1}$. As $k = n$, it is a volume form $C_M^n = |C_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|C_M^n|$ is a function on $M$.

One can also define $E_M^k$ and $C_M^k$ relative to a polydisc instead of a ball. They are different measures, but enjoy similar properties. In the sequel, we shall use $I_M^k$ to represent either $E_M^k$ or $C_M^k$.

The following theorem follows almost immediately from the definitions [4].
THEOREM (a). (1) $E^k_M \geq C^k_M$ on any complex manifold $M$.

(2) Let $f: M_1 \to M_2$ be a holomorphic mapping between complex manifolds $M_1$ and $M_2$. Then one has $I^k_{M_1} \geq f^*(I^k_{M_2})$, a measure-decreasing property under $f$.

(3) Let $X$ be a domain of a complex manifold $Y$. Then $I^k_X \geq I^k_Y$, a monotone property, holds.

(4) Any biholomorphism $f$ of a complex manifold $X$ is measure-preserving relative to $I^k_X$, that is, $I^k_X = f^*(I^k_X)$.

(5) Let $\widetilde{M}$ be a covering of a complex manifold $M$. Denote $\pi: \widetilde{M} \to M$ as the covering projection. Then $E^k_{\widetilde{M}} = \pi^*(E^k_M)$.

THEOREM (b) [3, 8]. Let $D$ be a bounded domain in $\mathbb{C}^n$ with a strongly pseudoconvex boundary point $p \in \partial D$. We denote $\tilde{D} = V \cap D$, where $p \in V$ is a sufficiently small ball in $\mathbb{C}^n$. Then the following is true: $|E^0_D(z)|/|C^0_D(z)|$ approaches one as $z \to p$.

In [7], the next theorem was proved for the special case where $D$ is completely hyperbolic. Actually, a similar proof can yield a slightly more general statement as follows.

THEOREM (c) [7]. Let $D$ be a bounded domain in $\mathbb{C}^n$. Suppose that there is one point $x \in D$ such that $|E^0_D(x)| = |C^0_D(x)|$. Then $D$ is biholomorphic to the euclidean ball.

THEOREM (d) [2] (CARTAN'S FIXED POINT THEOREM). Let $(X, ds^2)$ be a simply-connected complete Riemannian manifold with nonpositive sectional curvature. Suppose $G$ is a compact Lie group acting on $X$ as isometries. Then $G$ has a fixed point.

In particular, any finite group $H$ acting on $X$ isometrically must fix at least one point.

THEOREM (e). Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$. Suppose that

(1) there is a strongly pseudoconvex point $p \in \partial D_2$;

(2) one can find $x \in D_1$ and a sequence of holomorphic mappings $\{f_j\} \subset \text{Hol}(D_1, D_2)$ such that $\{f_j(x)\} \to p$.

Then there exists a subsequence of $\{f_j\}$, denoted by the same notation $\{f_j\}$, satisfying the property: For any compact set $K \subset D_1$ and any open set $\tilde{D} = V \cap D_2$, where $p \in V$ is an open set in $\mathbb{C}^n$, there is a $j_0$ in such a way that $f_j(K) \subset \tilde{D}$ for all $j \geq j_0$.

PROOF. Since $\{f_j(x)\} \to p$, by normal family argument one can find a subsequence of $\{f_j\}$ converging on compacta to a holomorphic mapping $f: D_1 \to \mathbb{C}^n$ so that $f(x) = p$ and $f(D_1) \subseteq \partial D_2$. By assumption, $\partial D_2$ is strongly pseudoconvex at $p$ and it contains no complex analytic variety of positive dimension through $p$. This implies $f$ is a constant mapping which brings the whole $D$ onto a single point. Our claim in Theorem (e) should now be clear.

(B) Proof. Let us assume $|E^0_{D_1}(x)| = |C^0_{D_1}(x)|$ for the given point $x$ in $D_1$. By Theorem (c), this implies that $D_1$ must be biholomorphic to $B_n$. If the order of the covering $f_j: B_n = D_1 \to D_2$ is greater than one, this would contradict Cartan’s fixed point theorem (Theorem (c)) because the Bergman metric on $B_n$ has
negative sectional curvature and it is invariant under biholomorphisms. Thus $D_2$ is also biholomorphic to $B_n$. Therefore, the whole proof depends on the following assertion.

**Claim.** $|E^n_{B_1}(x)| = |C^n_{B_1}(x)|$.

**Proof.** For each $j$, $f_j : D_1 \rightarrow D_2$ is a covering. From Theorem (a)(5) we have $$E^n_{B_1}(x,v) = E^n_{D_2}(x_j, df_j(v)),$$
where $x_j = f_j(x)$ and $(x,v)$ is a nonzero $n$-vector at $x$. Let $(D_1)_k$ be an increasing sequence of domains such that $\bigcup_{k=1}^\infty (D_1)_k = D_1$, $x \in (D_1)_k$ for each $k$, and $(D_1)_k \subset (D_1)_{k+1}$. For each $j$, denote $(D_2)_k^j = f_j(D_1)_k$. For a fixed $k$, we obtain by Theorem (a)(2)(3) the inequalities

$$C^n_{(D_1)_k}(x,v) \geq C^n_{(D_1)_k^j}(x_j, df_j(v)) \geq C^n_{D}(x_j, df_j(v)).$$

The last inequality on the above chain is valid for sufficiently large $j$. The reason is that when $j$ is sufficiently large, $f_j((D_1)_k) = (D_2)_k^j \subset \tilde{D}$ by Theorem (e), where $\tilde{D} = V \cap D_2$, $p \in V$ is an open set in $C^n$. It follows that for fixed $k$ and large $j$, we have the chain

$$C^n_{(D_1)_k}(x,v) \geq C^n_{(D_2)_k^j}(x_j, df_j(v)) \geq C^n_{D}(x_j, df_j(v))$$

of inequalities (Theorem (a)(5) has been used here).

Observe that:

(i) By the volume decreasing property under holomorphic mappings (Theorem (a)(2)), we have $E^n_{D}(x_j, df_j(v)) \geq E^n_{D_2}(x_j, df_j(v))$ as the inclusion map $\tilde{D} \hookrightarrow D_2$ is holomorphic. Therefore, we have

$$C^n_{(D_1)_k}(x,v) \geq C^n_{D}(x_j, df_j(v)).$$

(ii) Again by the strong pseudoconvexity of $p \in \partial D_2$, one obtains

$$C^n_{D}(x_j, df_j(v)) \rightarrow 1 \quad \text{as} \quad x_j \rightarrow p$$

by Theorem (b).

(iii) If we let $k \rightarrow \infty$, then $C^n_{(D_1)_k}(x,v) \rightarrow C^n_{(D_1)}(x,v)$. This approximation property can be proved by elementary normal family argument.

(iv) It is always true that $C^n_{(D_1)(x,v)}/E^n_{(D_1)}(x,v) \leq 1$ by Theorem (a)(1). Combining (i)–(iv), and letting $j \rightarrow \infty$ and then $k \rightarrow \infty$, one concludes that $1 \geq C^n_{(D_1)}(x,v)/E^n_{(D_1)}(x,v) \geq 1$, proving our claim.

**BIBLIOGRAPHY**


8. ——, *Estimate of intrinsic measures on s.p.c. domains*, unpublished manuscript.

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