Q-SETS DO NOT NECESSARILY HAVE STRONG MEASURE ZERO

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ABSTRACT. The purpose of this paper is to give a negative answer to the following question (see Miller [4]): Do all Q-sets have strong measure zero?

1. Definitions and standard facts.

1.1 Q-set. A set of reals X is a Q-set iff every subset of X is a relative $F_\sigma$. The history of Q-sets can be found in Fleissner’s paper [2]. We recall the following facts:

(i) If X is a Q-set then $|X| < 2^{R_\omega}$ and $2^{|X|} = 2^{R_\omega} = c$.

(ii) Every Q-set has universal measure zero.

(iii) Martin’s axiom implies that if $X \subseteq R$ and $|A| < 2^{R_\omega}$, then A is a Q-set.

1.2 Strong measure zero set. A set of reals X has strong measure zero iff given any sequence $\varepsilon_n > 0$ for $n < \omega$, X can be covered by a sequence of open sets $X_n$ each having diameter less than $\varepsilon_n$.

1.3 Ramsey ultrafilters. An ultrafilter $U \subseteq P(\omega)$ is a Ramsey ultrafilter iff $U$ contains the filter of cofinite sets and for any $\pi: [\omega]^2 \rightarrow 2$ there is an $A \in U$ with $\pi$ constant on $[A]^2$. For $A, B$ subsets of $\omega$, we say that $A \subseteq^* B$ iff there exists $n \in \omega$ such that $A - n \subseteq B$.

We say that a family $(A_\alpha: \alpha < \kappa)$, $\kappa$ a cardinal, is a tower iff $A_\beta \subseteq^* A_\alpha$ and $A_\alpha \nsubseteq^* A_\beta$ for every $\alpha < \beta$, and for every $A \subseteq \omega$, it is not the case that $\forall \alpha < \kappa$ $A \subseteq^* A_\alpha$.

The following facts are well known:

(i) Martin’s axiom implies $\kappa = 2^{R_\omega}$.

(ii) Martin’s axiom implies that there exists a Ramsey ultrafilter which is generated by a tower.

Let $U$ be a Ramsey ultrafilter over $\omega$. We define the following poset $P_U$: the elements of $P_U$ are ordered pairs $(s, A)$ such that $s \in \omega^{<\omega}$, $A \in U$, $\sup s < \inf A$, and the order is given by: $(s, A) \leq (t, B)$ iff

$s \subseteq t$, $B \subseteq A$ and $t - s \subseteq A$.

It is clear that $P_U$ satisfies the countable chain condition and the generic object can be regarded as a subset of $\omega$ characterized by being almost contained in every member of the filter $U$ (see Mathias [5]).

2. Theorem. Let $V$ be a model for ZFC+Martin’s axiom, let $U \subseteq V$ be a Ramsey ultrafilter generated by a tower $(A_\alpha: \alpha < c)$, let $P_U$ be the forcing notion defined above this $U$, and let $G \subseteq P_U$ be a generic object over $V$. Then

(i) $V$ and $V[G]$ have the same cardinals.

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(ii) $V[G] \models \text{"}c = c^V\text{"}$. 

(iii) If $X \in V \cap P(\mathbb{R})$ and $|X| < c$, then 
$V[G] \models \text{"}X \text{ is a } \mathcal{Q}\text{-set}\text{"}$. 

(iv) If $X \in V \cap P(\mathbb{R})$ and $|X| > \aleph_0$, then 
$V[G] \models \text{"}X \text{ does not have strong measure zero}\text{"}$. 

2.1 REMARK. In $V[G]$, the old uncountable subsets of reals of cardinality less than $c$, are $\mathcal{Q}$-sets but not of strong measure zero.

PROOF. Clear by (iii) and (iv).

2.2 Proof of the theorem. (i) By countable chain condition of $P_U$.

(ii) By countable chain condition every real in $V[G]$ is obtained by a name which is encodable in $V$ by a real.

(iii) Let $X \in V \cap P(\mathbb{R})$ and $|X| < c$. Let $h : X \to \{0,1\}$ be a $P_U$-name for a subset of $X$. By Mathias [5], for every $i \in X$ there exists $A_i \in U$ such that if $n \in A_i$ and $s \subseteq n$, then 
$$(s, A_i - n) \models \text{h}(i) = 0 \quad \text{or} \quad (s, A_i - n) \not\models \text{h}(i) = 1.$$ 
Since $U$ is generated by a tower, and $|X| < c$, there exists $A \in U$ such that for every $i \in X$, $A \subseteq^* A_i$. Therefore, for every $i \in X$ there exists $n_i \in \omega$ such that $A - n_i \subseteq A_i$ and $n_i \in A_i$.

So if $(\phi, A) \in G$, and $r (\subseteq \omega)$ is the real number defined by $G$, we have that $\text{h}(i)$ is computable from $r \upharpoonright n_i$.

Now we define the following equivalence relation on $X$:

$$i \sim j \iff n_i = n_j \quad \text{and} \quad (\forall s \subseteq n_i)((s, A_i - n_i) \models \text{h}(i) = 0 \iff (s, A_j - n_j) \not\models \text{h}(j) = 0).$$

It is clear that $\sim$ is an equivalence relation with countably many classes, say $X = \bigcup_{i \in \omega} X_i$ where each $X_i$ is an equivalence class and the following holds:

if $i, j$ belong to $X_i$ for $l \in \omega$, then 
$$(\phi, A) \models \text{h}(i) = \text{h}(j).$$

Each $X_i$ for $l \in \omega$ belongs to $V$ and also $\langle X_i : l \in \omega \rangle$ is a number of $V$. Since $V \models \text{MA}$ for every $l \in \omega$, there exists $Y_l$, an $\mathcal{F}_\sigma$ set of reals, such that 
$$V \models X_l = Y_l \cap X.$$ 
Therefore, by an absoluteness argument,
$$V[G] \models X_l = Y_l \cap X$$
(remember that $Y_l$ is a definition of a set), and thus in $V[G]$
$$\{i : h(i) = 0\} = X \cap \left(\bigcup_{i \in X_l}(\forall i \in X_l)(h(i) = 0)\right),$$
and this says that $\{i : h(i) = 0\}$ is a $\mathcal{F}_\sigma$ set relative to $X$. This completes the proof of (iii).

(iv) This fact is well known and the proof is obtained following the argument given by Baumgartner [1, §9] in which it is possible to replace Mathias’ forcing by $P_U$ and to use the results proven by Mathias [5].

This concludes the proof of the theorem, and the following question arises: Is “ZFC+ Borel conjecture + there exists $\mathcal{Q}$-set” consistent?
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