MAXIMAL IDEALS IN THE BURNSIDE RING
OF A COMPACT LIE GROUP

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ABSTRACT. A new criterion is found for deciding whether or not two maximal ideals in the Burnside ring of a compact Lie group coincide. One consequence is that certain algebraic and topological localizations in equivariant stable homotopy theory are naturally isomorphic.

Let $G$ be a compact Lie group and $H$ a (closed) subgroup. For a prime $p$, let $q(H,p)$ be the kernel of the homomorphism $A(G)\to\mathbb{Z}_p$ obtained by sending the equivalence class of a finite $G$-CW complex $X$ to the mod $p$ reduction of the Euler characteristic $\chi(X^H)$. All maximal ideals of the Burnside ring $A(G)$ are of this general form. Our main goal is to obtain a satisfactory criterion for determining when $q(J,p) = q(H,p)$. The reader is referred to [1, 5.7 or 2, V, §3] for background and earlier results. We shall obtain the following result as a consequence of our criterion.

**Theorem 1.** If $H \subset J \subset K$ and $q(H,p) = q(K,p)$, then $q(J,p) = q(K,p)$.

This gives a positive answer to a question raised in [2, V.3.8]. As explained in detail in [2, V.8.9], the theorem has the following consequence in equivariant stable homotopy theory. Let $WH = NH/H$.

**Theorem 2.** Let $q = q(K,p)$, where $|WK|$ is finite and prime to $p$, and let

$$\mathcal{E} = \{H \mid q(J,p) \neq q \text{ for all } J \subset H\},$$

$$\mathcal{F}' = \{H \mid (H) \leq (K)\} \quad \text{and} \quad \mathcal{F} = \mathcal{E} \cap \mathcal{F}'. $$

Then, for finite $G$-CW spectra $X$ and general $G$-spectra $Y$, the algebraic localization $[X,Y]^G_q$ is naturally isomorphic to the topological localization $[X,E(\mathcal{F}',\mathcal{F}) \wedge Y]^G_{(p)}$.

Here $E(\mathcal{F}',\mathcal{F})$ is a based $G$-CW complex characterized up to $G$-equivalence by the requirement that $E(\mathcal{F}',\mathcal{F})^H$ be equivalent to $S^0$ if $H \notin \mathcal{F}' - \mathcal{F}$ and be contractible otherwise (see e.g. [2, V, §7]). The importance of Theorem 2 is that $[X,Y]^G_{(p)}$ can be computed in terms of the various algebraic localizations $[X,Y]^G_q$ (by [2, V.5.5]). The relevance of Theorem 1 is that it implies

$$\mathcal{F}' - \mathcal{F} = \{H \mid q(H,p) = q(K,p)\}. $$

To see this, recall that, for any maximal ideal $q$ of residual characteristic $p$, there is one and, up to conjugacy, only one $K$ with $|WK|$ finite and prime to $p$ such that
q = q(K, p); moreover, this K is maximal in \{H|q(H, p) = q\} (see [1, 5.7.2 or 2, V.3.1]). For a given H, we let H^p denote a subgroup of G such that q(H, p) = q(H^p, p) and |WH^p| is finite and prime to p.

If G is finite and H_p is the smallest normal subgroup of H such that H/H_p is a p-group, then q(H, p) = q(J, p) if and only if H_p is conjugate to J_p. We wish to generalize this criterion to the compact Lie case. Say that a group is p-perfect if it admits no nontrivial quotient p-groups. Then H_p above can also be characterized as the maximal p-perfect subgroup of H. In the compact Lie case, we let H'_p denote the maximal p-perfect subgroup of H. It can be constructed explicitly as the inverse image in H of (H/H_0)_p \subset H/H_0, where H_0 is the component of the identity element. We then define H_p \subset NH'_p to be the inverse image of a maximal torus in WH'_p. H_p is still p-perfect, but now WH_p is finite [1, 5.7.5 or 2, V.3.3]. Note that q(H, p) = q(H'_p, p) = q(H_p, p) since H/H'_p is a p-group and H_p/H'_p is a torus [1, 5.7.1 or 2, V.3.6]. The following is our main result. We are indebted to the referee for its proper formulation.

**Theorem 3.** Let q = q(K, p), where |WK| is finite and prime to p, and let L = K_p.

(i) L = K'_p; that is, L is the maximal p-perfect subgroup of K; moreover, L is the unique normal p-perfect subgroup of K whose quotient is a finite p-group.

(ii) L is maximal in \{H | q(H, p) = q and H is p-perfect\}; up to conjugacy, this property uniquely characterizes L.

(iii) If H \subset G, then q(H, p) = q if and only if H_p is conjugate to L.

(iv) If H \subset L is p-perfect and q(H, p) = q, then HT = L, where T is the component of the identity element of the center of L.

Part (iii) can be restated as follows.

**Corollary 4.** q(H, p) = q(J, p) if and only if H_p is conjugate to J_p.

Parts (i) and (iv) imply Theorem 1.

**Proof of Theorem 1.** We are given H \subset J \subset K and q(H, p) = q(K, p). Expanding K if necessary, we may assume that |WK| is finite and prime to p. We have H'_p \subset J'_p \subset K'_p since passage to maximal p-perfect subgroups preserves inclusions. By (i), K'_p = L. By (iv), H'_pT = L. Therefore J'_pT = L. Since J'_pT/J'_p is a torus,

\[
q(J, p) = q(J'_p, p) = q(J'_pT, p) = q(K, p).
\]

The proof of Theorem 3 depends on the following observation.

**Proposition 5.** If H is p-perfect and WH is finite, then, up to conjugation, H is a normal subgroup of H^p with quotient a finite p-group.

**Proof.** The argument is identical to the proof of [1, 5.7.8 or 2, V.3.6(iii)]], which give the same conclusion under the stronger hypothesis that |H/H_0| \neq 0 mod p rather than that H is p-perfect.

**Proof of Theorem 3.** (i) By the proposition, L is subconjugate to K. Since K/K'_p is finite and L/K'_p is a torus, this implies L = K'_p. By the maximality of K'_p, if H \subset K is p-perfect, then H \subset L. If, further, H \triangleleft K and K/H is a p-group, then H = L since L is p-perfect.
(ii) Since \( K \) is maximal in \( \{ H \mid q(H, p) = q \} \) and is characterized up to conjugacy by this property, (ii) follows immediately from (i).

(iii) Since \( q(H, p) = q(H_p, p) \), sufficiency is clear. Applying the proposition to \( H_p \) and then applying part (i) to \( H^p \), we see that \( H_p \) is conjugate to \( (H^p)_p \). This implies necessity.

(iv) Since \( H \) is \( p \)-perfect and \( H < HT \) with quotient a torus, the construction of \( H_p \) shows that we may assume \( HT \subset H_p \). Since \( H_p/H \) is a torus, \( HT \subset H_p \) and \( H_p/HT \) is a torus. As a compact subgroup of the discrete group of automorphisms modulo inner automorphisms of \( HT \), \( H_p/SHT \) is finite and thus trivial, where \( S \) is the centralizer of \( HT \) in \( H_p \). Clearly \( S \) contains \( T \) and is the center of \( H_p \). Since \( q(H, p) = q \), \( H_p \) is conjugate to \( L \) by part (iii). Therefore the identity component of \( S \) is isomorphic to, and thus equal to, \( T \). It follows that \( H_p = HT \subset L \) and hence \( H_p = HT = L \).

REFERENCES
