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ON THE CARDINALITY OF A TOPOLOGY

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ABSTRACT. Let $o(X)$ denote the cardinality of topology of a space X. I. Juhasz proves that $o(X)^\omega = o(X)$ for regular hereditarily paracompact spaces. We prove it for more general classes of spaces.

If X is a topological space, let $o(X)$ denote the cardinality of its topology. A number of papers [**1**, **2**, **3**] have investigated classes for which $o(X) = o(X)^\omega$. We generalize several of these results by proving

THEOREM 1. *For every infinite hereditarily weakly θ-refinable T_3 space X, $o(X) = o(X)^\omega$.*

A space X is *weakly θ-refinable* [**4**] if and only if each open cover $\mathscr{U}$ of X has a refinement $\mathscr{V} = \bigcup_{n\in\omega}\mathscr{V}_n$ where each $\mathscr{V}_n$ is a discrete family of subsets of the space $\bigcup\mathscr{V}_n$.

If η and μ are infinite cardinals and $\mu' < \mu$ implies $\eta^{\mu'} < \eta^\mu$ and $\eta' < \eta$ implies $\eta'^\mu < \eta^\mu$, then η^μ is called a *jump*. By Theorem 0.2 of [**2**], if η^μ is a jump, μ is regular.

Since X is T_3, by Theorem 4.3 of [**2**], if μ is an uncountable cardinal of cofinality ω and there is a discrete subset S' of X with $|S'| = \mu'$ for all $\mu' < \mu$, then there is a discrete subset S of X with $|S| = \mu$.

Before proving the theorem, we also need

LEMMA. *If $\langle \kappa_\alpha \mid \alpha \in \gamma\rangle$ is a sequence of infinite cardinals, then there is an $S \subset \gamma$ with $\prod_{\alpha\in\gamma}\kappa_\alpha = (\sup_{\alpha\in S}\kappa_\alpha)^{|S|}$.*

PROOF OF LEMMA. Enumerate the κ_α so that they form a nondecreasing sequence. Let $S \subset \gamma$ be of minimal order type so that $\prod_{\alpha\in\gamma}\kappa_\alpha = \prod_{\alpha\in S}\kappa_\alpha$. Without loss of generality, suppose S is infinite. Note that if $\beta < \sup S$, then $S - (\beta + 1)$ has the order type of S, in particular $|S - (\beta + 1)| = |S|$. So if $|S| = \lambda$ we have that $S = \lambda \circ \alpha$ for $\alpha = 1$ or for some limit ordinal $\alpha < \lambda^+$. Split S into λ many cofinal disjoint sets $\{A_\beta \mid \beta \in \lambda\}$. Note that every A_β is cofinal in S and so

$$\prod_{\delta\in A_\beta}\kappa_\delta \geqslant \sup_{\delta\in A_\beta}\kappa_\delta = \sup_{\alpha\in S}\kappa_\alpha = \kappa.$$

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Hence

$$\kappa^\lambda \leqslant \prod_{\beta\in\lambda}\prod_{\delta\in A_\beta}\kappa_\delta = \prod_{\alpha\in S}\kappa_\alpha; \quad \text{so } \kappa^\lambda = \prod_{\alpha\in\gamma}\kappa_\alpha. \qquad \square$$

PROOF OF THEOREM. Assume not and let X be a counterexample; so $o(X) = \kappa < \kappa^\omega$. Let $\lambda = \min\{\mu \mid \mu^\omega > \kappa\}$. Then $\mathrm{cf}(\lambda) = \omega$ and $\lambda \leqslant \kappa$.

For $x \in X$, let $\sigma(x, X) = \min\{o(U) \mid x \in U \text{ and } U \text{ is open}\}$ and $\sigma = \sigma(X) = \sup\{\sigma(x, X) \mid x \in X\}$.

Note that there exists a finite subset $F \subset X$ such that $\sigma(X - F) < \lambda$. To see this assume the contrary and let $\langle \lambda_n \mid n \in \omega\rangle$ be a cofinal sequence in λ. At stage n pick $x_n \in X - \{x_k \mid k \in n\}$ with $\sigma(x_n, X) \geqslant \lambda_n$. Since X is regular there is a disjoint open family $\{U_n \mid n \in \omega\}$ with $x_n \in U_n$, hence $o(U_n) \geqslant \lambda_n$. So $o(X) \geqslant \prod_{n\in\omega} o(U_n) = \lambda^\omega > \kappa$, a contradiction.

Thus we can assume that $\sigma(X) < \lambda$ and that $o(X) = o(X - F)$ for all finite $F \subset X$ (recall that our original X was *hereditarily* weakly θ-refinable).

Let $\mathscr{U}$ be an open cover of X such that each U in $\mathscr{U}$ has $o(U) \leqslant \sigma$. Since X is weakly θ-refinable there is a refinement $\mathscr{V} = \bigcup_{n\in\omega}\mathscr{V}_n$ such that each $\mathscr{V}_n$ is a closed discrete family and $\bigcup\mathscr{V}_n = V_n$. Assume $\mathscr{V}_n = \{V_n^\alpha \mid \alpha \in \beta_n\}$ and β_n is infinite for each $n \in \omega$. Then $\kappa = o(X) \leqslant \prod_{n\in\omega} o(V_n)$. Now we derive a contradiction by showing that $\prod_{n\in\omega} o(V_n) < \kappa$.

Fixing $n \in \omega$, we have $o(V_n) = \prod_{\alpha\in\beta_n} o(V_n^\alpha) = \sigma_n^{\mu_n}$ for $\sigma_n = \sup\{o(V_n^\alpha) \mid \alpha \in \beta_n\}$ and $\mu_n = |\beta_n|$ by the lemma. Since $o(V_n^\alpha) \leqslant \sigma$ for every α, we have that $\sigma_n \leqslant \sigma$. Also $\sigma_n^{\mu_n} < \lambda$ since $o(V_n) \leqslant o(X) = \kappa$ and $\sigma_n^{\mu_n\cdot\omega} = \sigma_n^{\mu_n} \leqslant \kappa$ if $\mu_n \geqslant \omega$; otherwise $\sigma_n^{\mu_n} \leqslant \sigma < \lambda$. Note that $\prod\{\sigma_n^{\mu_n} \mid n \in \omega \text{ and } \sigma_n^{\mu_n} < \sigma\} \leqslant \sigma^\omega < \lambda$. So in order to show that $\prod_{n\in\omega}\sigma_n^{\mu_n} < \kappa$ we may assume that $\sigma \leqslant \sigma_n^{\mu_n}$ for all n. Then $\sigma_n^{\mu_n} = \sigma^{\mu_n} < \lambda$ (since $\sigma_n \leqslant \sigma \leqslant \sigma_n^{\mu_n} < \lambda$). Hence if $\mu = \sup_{n\in\omega}\mu_n$, $\prod_{n\in\omega}\sigma_n^{\mu_n} = \sigma^\mu$. If $\sigma^\mu < \lambda$, then $\prod_{n\in\omega} o(\mathscr{V}_n) < \kappa$ as desired.

So assume that $\sigma^\mu \geqslant \lambda$. Since $\sigma^\omega < \lambda$, μ is uncountable and, since $\sigma^{\mu_n} < \lambda$ for all n, $\mathrm{cf}(\mu) = \omega$. If $\eta = \min\{\alpha \mid \alpha^\mu \geqslant \lambda\}$, then $\eta' < \eta$ implies $\eta'^\mu < \eta^\mu$ and $\mu' < \mu$ implies $\eta^{\mu'} < \eta^\mu$. So if η is infinite, η^μ is a jump and μ is regular. This contradicts μ being uncountable and of cofinality ω.

So it remains to show that η is infinite if μ is. To show this, for each $n \in \omega$ and $\alpha \in \beta_n$ choose $x_\alpha \in V_n^\alpha$ and let $S_n = \{x_\alpha \mid \alpha \in \beta_n\}$. Since S_n is discrete and $|S_n| = \mu_n$, there is a discrete $S \subset X$ of cardinality μ. Thus $2^\mu \leqslant o(S) \leqslant o(X) = \kappa$. Since μ is infinite, $2^{\mu\cdot\omega} = 2^\mu \leqslant \kappa$ while $\lambda^\omega > \kappa$, so $2^\mu < \lambda$. Since $2^\mu = \eta^\mu$ for finite η and $\eta^\mu \geqslant \lambda$ by definition, η is infinite. $\square$

Our proof gives the stronger

THEOREM 2. *Let X be regular space. If for any finite subset F, $X - F$ is weakly θ-refinable, then $o(X) = o(X)^\omega$.*

Since weak θ-refinability is hereditary to F_σ-sets, $o(X) = o(X)^\omega$ also holds for 1st countable, T_3, weakly θ-refinable spaces or 1st countable, T_3 paracompact spaces.

A similar proof also yields a theorem, independently proved by I. Juhasz, that $o(X) = o(X)^\omega$ for all hereditarily weakly collectionwise Hausdorff spaces.

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