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ON DENDROIDS WITH KELLEY'S PROPERTY

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ABSTRACT. It is proved that if a dendroid has Kelley's property, then it is smooth. This is a correction of an error from [**4**].

S. B. Nadler, Jr., in Example (5.10) on p. 258 of his book [**4**] says that there is a dendroid which is not contractible so it is not smooth but it has Kelley's property (property K). This example does not have property K, contrary to assertion. In this paper we show that there cannot exist a continuum with the properties claimed by Nadler. Namely, it is proved that if a dendroid has Kelley's property, then it is smooth.

A continuum is a compact connected metric space. A continuum X is called hereditarily unicoherent if for any two subcontinua A and B of X the intersection $A \cap B$ is connected. A dendroid is an arcwise connected and hereditarily unicoherent continuum.

We say that a continuum X is smooth at the point $p \in X$ (see [**3**, p. 81]) if for each sequence x_n convergent to some $x \in X$ and for each subcontinuum K of X such that $p, x \in K$, there exists a sequence K_n of subcontinua of X such that $p, x_n \in K_n$ and $\operatorname{Lt} K_n = K$. A continuum is smooth if it is smooth at some point.

Given a point a of a continuum X, we define $T(a)$ as the set of all points $x \in X$ such that every subcontinuum of X which contains x in its interior must contain a.

If A and B are subcontinua of a metric continuum X with a metric d, then define $H(A, B) = \max\{\sup d(a, B), \sup d(b, A)\}$. It is known (see [**4**, Theorem (0.2), p. 2, and Remark (0.4), p. 3]) that H is a metric in $C(X)$—the space of all subcontinua of a continuum X.

A continuum X is said to have property K (see [**4**, Definition (16.10), p. 538]), provided that given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a, b \in X$, $d(a, b) < \delta$ and $a \in A \in C(X)$, then there exists $B \in C(X)$ such that $b \in B$ and $H(A, B) < \varepsilon$.

An arc with endpoints a and b will be denoted by ab. The arc ab will be, in some cases, regarded as an ordered arc with a as the first and b as the last point. An order will be denoted by the symbol $<$. Let ab be an arc in a space X and let $U_1, U_2, \dots$ be a sequence of subsets of X. We say that the ordered arc $ab \subset X$ has type $(U_1, U_2, \dots)$ (see [**2**, p. 228]), and write $ab \in (U_1, U_2, \dots)$ if there exists a sequence of points $a_1, a_2, \dots$ satisfying the conditions

(i) $a_n \in ab \cap U_n$ for each $n \geq 1$,

(ii) $a < a_1 < a_2 \cdots < b$.

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LEMMA 1. *For each pair of points x and y of a dendroid X we have that $y \in T(x)$ if and only if there exists a sequence y_n of points of X converging to y such that $x \in \operatorname{Lt} y_n y$.*

PROOF. Suppose that there is a sequence y_n convergent to y such that $x \in \operatorname{Ls} y_n y$. Let K be a subcontinuum of X such that $y \in \operatorname{Int} K$. Then $y_n \in K$ and $y_n y \subset K$. Therefore $\operatorname{Ls} y_n y \subset K$, so $x \in K$ and thus $y \in T(x)$. The inverse implication is evident.

We say that the dendroid X has property $(*)$ if for each pair of points x and y of X we have that if $T(x) \cap xy \neq \{x\}$, then $y \in T(x)$.

LEMMA 2. *If a dendroid X has property* K, *then X has property* $(*)$.

PROOF. Let a dendroid X have property K and let x and y be points of X such that $T(x) \cap xy \neq \{x\}$. Let $z \in (T(x) \cap xy) - \{x\}$. By Lemma 1 we have a sequence z_n of points of X converging to z such that $x \in \operatorname{Lt} z_n z$. Put $\varepsilon_k = 1/k$. By property K there are δ_k and a subsequence z_{n_k} such that $d(z, z_{n_k}) < \delta_k$. Thereby there are subcontinua B_k of X such that $z_{n_k} \in B_k$ and $H(B_k, zy) < \varepsilon_k$. From this we have a sequence y_k of points of X converging to y such that $y_k \in B_k$. So each subcontinuum containing y in its interior must contain y_k for almost all k. Thus by hereditary unicoherence of X we have that $y \in T(x)$. The proof is complete.

PROPOSITION 3 (see [**1**, Theorem 6, p. 302]). *A dendroid X is smooth if and only if for each pair of points $x, y \in X$ either $xy \cap T(x) = \{x\}$ or $xy \cap T(y) = \{y\}$.*

THEOREM 4. *If a dendroid X has property $(*)$, then X is smooth.*

PROOF. Suppose that the dendroid X is not smooth. Therefore, by Proposition 3, we have that there exist points x and y of X such that $T(x) \cap xy \neq \{x\}$ and $T(y) \cap xy \neq \{y\}$. By property $(*)$ we have that $y \in T(x)$ and $x \in T(y)$ so by Lemma 1 there exists a sequence x'_n converging to x such that $y \in \operatorname{Lt} x'_n x$. Let us denote by U and V open subsets of X such that $x \in U$, $y \in V$ and $\operatorname{Cl}(U) \cap \operatorname{Cl}(V) = \varnothing$. It is easy to see that for almost all n, arcs $x'_n x \in (U, V, U)$. Without loss of generality we can assume that

$$\text{for all } n \text{ we have that } x'_n x \in (U, V, U). \tag{1}$$

Case 1. There is a subsequence $\{x'_{n_k}\}$ of the sequence $\{x'_n\}$ such that $x'_{n_k} x \cap xy = \{x\}$. Thus, by $(*)$, $x'_{n_k} \in T(y)$ and there are points $z_m(1, n_k)$ such that the sequence $\{z_m(1, n_k)\}$ is convergent to the point x'_{n_k} and $y \in \operatorname{Lt} x'_{n_k} z_m(1, n_k)$. By (1) we have that, for almost all m, $x z_m(1, n_k) \in (U, V, U, V, U)$. We can find a subsequence $\{x''_k\}$ of the sequence $\{z_m(1, n_k)\}$ which is convergent to x. For this sequence we have that $x''_k x \in (U, V, U, V, U)$.

Case 2. For almost all n, there is $x'_n x \cap xy \neq \{x\}$. Thus, by $(*)$, $x'_n \in T(x)$, and there are points $w_m(1, n)$ such that the sequence $\{w_m(1, n)\}$ is convergent to x'_n and $x x'_n \subset \operatorname{Lt} w_m(1, n) x$. It follows from (1) that for almost all m, $w_m(1, n) y \in (U, V, U, V)$. We can find a subsequence $\{w'_k\}$ of the sequence $\{w_m(1, n)\}$ such that w'_k is convergent to x and $w'_k y \in (U, V, U, V)$ for all $k \in N$. Now, once more by $(*)$, $w'_k \in T(y)$ and there are points $v_s(1, n, m)$ such that the sequence $\{v_s(1, n, m)\}$ is convergent to w'_k and $x w'_k \subset \operatorname{Lt} v_s(1, n, m) w'_k$. We can easily see that $x v_s(1, n, m) \in (U, V, U, V, U)$ for almost all $s \in N$. So, we can find a subsequence $\{x''_r\}$ of the sequence $\{v_s(1, n, m)\}$ which is convergent to x and $x x''_r \in (U, V, U, V, U)$.

In the same way we construct a sequence $\{x_n'''\}$ such that $x_n''' \in (U, V, U, V, U, V, U)$, and so on. Thus it is evident that the dendroid X contains a subcontinuum Y which is homeomorphic to the sin-curve $S = \{(x, y) \colon y = \sin 1/x \text{ for } 0 < x \leq 1 \text{ and } y \in [-1, 1] \text{ for } x = 0\}$. This is a contradiction, so the proof is complete.

COROLLARY 5. *If a dendroid has property* K, *then it is smooth.*

Let us observe that if a continuum X is not a dendroid, then smoothness does not follow from property K.

EXAMPLE 6. Let C denote a Cantor set and let $Y = S \times C$. Denote by f a map of $S \times C$ which for each $(0, y) \in S$ identifies the points $((0, y), c)$ and which is a homeomorphism on each set $S \times \{c\}$. Put $X = f(S \times C)$. It is easy to see that X has property K and that X is locally connected at no point $x \in X$, so X is not a smooth continuum.

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