PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 3, March 1988

RIGID STONE SPACES WITHIN ZFC

A. DOW, A. V. GUBBI AND A. SZYMAŃSKI

(Communicated by Dennis Burke)

ABSTRACT. We construct within ZFC a family of pairwise nonhomeomorphic dense-in-itself rigid extremally disconnected compact separable spaces. Any such space is determined by a sequence of weak P-points of $\mathbf{N}^*$ having different types.

Introduction. In their review paper, van Douwen, Monk and Rubin [**DMR**] state that examples of rigid Boolean algebras published in the literature are rather special and are not easily described. They also ask for some "natural" examples. In this paper the authors have constructed within ZFC a large family of separable, compact Hausdorff, extremally disconnected perfect (no isolated points) rigid topological spaces. Thanks to the duality theory of Stone, this translates in Boolean algebra to a family of nonatomic *complete* rigid Boolean algebras. Moreover, their structure is so simple and natural, as to be deemed canonical examples. We hope that, with these examples, we have answered the question of [**DMR**].

Our starting point is a known technique of construction of a family of Tychonov rigid spaces. (See for example Gubbi [**G**], or Kannan and Rajagopalan [**KR**].) Although by a Stone space is usually meant a compact Hausdorff totally disconnected space, for want of a suitable short word, we mean in the sequel an extremally disconnected (E.D.) compact Hausdorff space. A topological space is rigid if its only autohomeomorphism is the identity. All other terms are standard and can be easily located in standard texts.

Let Seq be the set of all finite sequences of natural numbers, $\mathbf{N}$. For every $s \in \mathrm{Seq}$ let ξ_s be a filter on $\mathbf{N}$ containing the Fréchet filter on $\mathbf{N}$. Define a topology $\mathcal{T}$ on Seq by the rule: $V \subset \mathrm{Seq}$ is open iff for every $s \in V$, the set $\{n\colon s^\frown n \in V\} \in \xi_s$. Here $s^\frown n$ denotes the concatenation of s by n [**J**]. It is easy to verify that Seq with this topology is Tychonoff, 0-dimensional and perfect. If ξ_s is the Fréchet filter for every $s \in \mathrm{Seq}$, then Seq with this topology is the space S_ω of [**AF**]. If ξ_s is an ultrafilter for every $s \in \mathrm{Seq}$ then Seq with $\mathcal{T}$ is an E.D. space.

In order to facilitate further discussion, let us consider some special subspaces of Seq. Let L_n denote the set of all sequences of length n (this is also the nth level of the tree of all sequences) and let T_n stand for the subspace of all sequences of length $\leq n$, $n \in \omega$. The sequence of length 0 is denoted s_0, the base point for the space constructed above. Observe that T_n is a closed nowhere dense subspace

Received by the editors June 13, 1986 and, in revised form, December 1, 1986. This paper was presented to the Annual Conference, AMS, San Antonio, Texas, in January 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54G05; Secondary 06E15.

Key words and phrases. Rigid space, weak P-point, $\mathbf{N}^*$.

The third author expresses his gratitude to the Department of Mathematical and Computer Sciences of the Youngstown State University for its hospitality during the preparation of this work.

of Seq and L_n is discrete and dense in T_n. We shall denote by $\mathcal{G}_\omega$ the space $(\mathrm{Seq}, \mathcal{T})$ determined by choosing ξ_s's to be free ultrafilters on ω. The following fact concerning E. D. Hausdorff spaces is quite obvious: if X is an E.D. Hausdorff space, A and B are σ-compact subsets of X such that $\mathrm{cl}\, A \cap B = \varnothing = A \cap \mathrm{cl}\, B$, i.e., A and B are separated, then $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \varnothing$. We shall now prove some facts concerning the separability properties of $\mathcal{G}_\omega^* = \beta\mathcal{G}_\omega - \mathcal{G}_\omega$, $\beta\mathcal{G}_\omega$ standing for the Stone-Čech compactification of $\mathcal{G}_\omega$.

LEMMA 1. *Let $U_1, U_2, \ldots$ be a sequence of clopen subsets of $\beta\mathcal{G}_\omega$ such that* (1) *$s_0 \notin U_1$, and* (2) *$U_n \cap ((L_1 - U_1) \cup (L_2 - U_2) \cup \cdots \cup (L_{n-1} - U_{n-1})) = \varnothing$ for $n > 1$. Then $s_0 \notin \mathrm{cl}(U_1 \cup U_2 \cup \cdots)$.*

PROOF. It is enough to prove that $V = \mathcal{G}_\omega - (U_1 \cup U_2 \cup \cdots)$ is a neighborhood of s_0 in $\mathcal{G}_\omega$. First, note that $s_0 \in V$. Let now $t \in V$ and let us suppose that the length of t is $m \geq 0$.

Since $t \in V$, $t \notin U_1 \cup \cdots \cup U_{m+1}$. Therefore $\mathcal{G}_\omega - (U_1 \cup \cdots \cup U_{m+1})$ is an open neighborhood of t. Hence $A = \{n \in \mathrm{N}\colon\ t^\frown n \in \mathcal{G}_\omega - (U_1 \cup \cdots \cup U_{m+1})\} \in \xi_t$. We shall show that for any $n \in A$, $t^\frown n \in V$. If not, then $t^\frown n \in U_k$ for some k, $k > m+1$. By assumption, $U_k \cap ((L_1 - U_1) \cup \cdots \cup (L_{k-1} - U_{k-1})) = \varnothing$. In particular, $U_k \cap (L_{m+1} - U_{m+1}) = \varnothing$. But if $t \in L_m$, then $t^\frown n \in L_{m+1}$, so $t^\frown n \in L_{m+1} - U_{m+1}$ for $n \in A$. Hence $U_k \cap (L_{m+1} - U_{m+1}) \neq \varnothing$; a contradiction.

LEMMA 2. *Let K be a σ-compact subset of βG_ω such that $K \cap \mathrm{cl}\ L_n = \varnothing$ for every n. Then $\mathrm{cl}\, K$ is contained in $\mathcal{G}_\omega^*$.*

PROOF. Let $K = \bigcup K_n$, K_n being compact. Let U_n be a clopen neighborhood of $K_1 \cup \cdots \cup K_n$ disjoint with $\mathrm{cl}\, L_1 \cup \cdots \cup \mathrm{cl}\, L_n$. We claim that $U_1, U_2, \ldots$ satisfy Lemma 1.

Indeed $U_1 \cap \mathrm{cl}\, L_1 = \varnothing$ and $s_0 \in \mathrm{cl}\, L_1$. Now if $n > 1$, then

$$U_n \cap ((L_1 - U_1) \cup \cdots \cup (L_{n-1} - U_{n-1})) \subset U_n \cap (\mathrm{cl}\, L_1 \cup \cdots \cup \mathrm{cl}\, L_{n-1}) = \varnothing.$$

By Lemma 1 $s_0 \notin \mathrm{cl}(U_1 \cup U_2 \cup \cdots)$. Hence $s_0 \notin \mathrm{cl}\, K$. A similar proof works for every $s \in \mathcal{G}_\omega$, and this completes the proof. □

A point of a space is said to be a *weak P-point* if it is not an accumulation point of any countable subset of the space.

THEOREM 1. *Suppose that for every $t \in \mathcal{G}_\omega$, ξ_t is chosen to be a weak P-point of N^*. Then the closure of every countable subset of $\mathcal{G}_\omega^*$ is contained in $\mathcal{G}_\omega^*$.*

PROOF. Let D be a countable subset of $\mathcal{G}_\omega^*$. Put $D_1 = \mathrm{cl}\, L_1 \cap D$ and $D_{n+1} = \mathrm{cl}\, L_{n+1} \cap (D - (D_1 \cup \cdots \cup D_n))$ for every n. Note that D_{n+1} is disjoint with $\mathrm{cl}\, L_n$. Because ξ_{s_0} is a weak P-point of N^*, $s_0 \notin \mathrm{cl}\, D_1$. Let U_1 be a clopen set in $B\mathcal{G}_\omega$, such that $U_1 \supset D_1$ and $s_0 \notin U_1$. Suppose we have defined $U_1, U_2, \ldots, U_n$ satisfying:

(1) $D_k \subset U_k$,

(2) $U_k \cap [\mathrm{cl}(L_1 - U_1) \cup \cdots \cup \mathrm{cl}(L_{k-1} - U_{k-1})] = \varnothing$

for $k = 2, \ldots, n$.

Now observe that $\mathrm{cl}\, D_{n+1} \cap [\mathrm{cl}(L_1 - U_1) \cup \cdots \cup \mathrm{cl}(L_n - U_n)] = \varnothing$. In order to see this, note first that since $D_{n+1} \subset \mathrm{cl}\, L_{n+1}$, $\mathrm{cl}\, D_{n+1} \cap L_{n+2} = \varnothing$. For every $s \in L_{n+1}$ the set $\{s^\frown n\colon\ n \in \mathrm{N}\}$ is a discrete subspace of $\mathcal{G}_\omega$. Since ξ_s is a weak P-point of $\beta\mathrm{N}$ and the induced topology on $\mathrm{cl}\{s^\frown n\colon\ n \in \mathrm{N}\}$ coincides with that for $\beta\mathrm{N}$ (by identifying n with $s^\frown n$), there is an $A \in \xi_s$ such that $\mathrm{cl}\{s^\frown n\colon n \in A\} \cap D_{n+1} = \varnothing$. Hence $\mathrm{cl}\{s^\frown n\colon n \in A\} \cap \mathrm{cl}\, D_{n+1} = \varnothing$, D_{n+1} and $\{s^\frown n\colon n \in A\}$ being two countable separated subsets of $\beta\mathcal{G}_\omega$. In consequence $s \notin \mathrm{cl}\, D_{n+1}$ for every $s \in L_{n+1}$, i.e., $\mathrm{cl}\, D_{n+1} \cap L_{n+1} = \varnothing$. Arguing as above for every $t \in L_n$, we get that $\mathrm{cl}\, D_{n+1} \cap L_n = \varnothing$. At the beginning of the proof we have noted that $D_{n+1} \cap \mathrm{cl}\, L_n = \varnothing$. Hence $\mathrm{cl}\, D_{n+1} \cap \mathrm{cl}\, L_n = \varnothing$, D_{n+1} and L_n being two separated subsets of $\beta\mathcal{G}_\omega$. Now we can find a clopen set $U_{n+1} \supset D_{n+1}$ such that $U_{n+1} \cap [\mathrm{cl}(L_1 - U_1) \cup \cdots \cup \mathrm{cl}(L_n - U_n)] = \varnothing$. In fact, any clopen set containing D_{n+1} and disjoint with L_n will possess this property. Using Lemma 1, $s_0 \notin \mathrm{cl}(U_1 \cup U_2 \cup \cdots)$. But $U_1 \cup U_2 \cup \cdots$ contains $D_1 \cup D_2 \cup \cdots = D \cap \bigcup\{\mathrm{cl}\, L_n\colon n \in \mathbf{N}\}$. In virtue of Lemma 2, $s_0 \notin \mathrm{cl}(D - \bigcup\{\mathrm{cl}\, L_n\colon n \in \mathbf{N}\})$. Therefore $s_0 \notin \mathrm{cl}\, D$. This proof can be repeated for every $t \in \mathcal{G}_\omega$, so the proof of the theorem is complete. □

REMARK 1. If we choose P-points instead of weak P-points, then one can extend Theorem 1 to σ-compact subsets of $\mathcal{G}_\omega^*$.

REMARK 2. Theorem 1 says that for a suitable choice of ultrafilters ξ_s, the space $\mathcal{G}_\omega^*$ is nowhere separable. The question about nonseparability of $\mathcal{G}_\omega^*$ was communicated to the second author by M. Rajagopalan.

Two ultrafilters $\xi_1, \xi_2 \in \beta\mathbf{N}$ are of the same type if there is a permutation f of $\mathbf{N}$ such that the extension of f to $\beta\mathbf{N}$ carries ξ_1 onto ξ_2.

THEOREM 2. *Suppose that ξ_t is chosen to be a weak P-point of $\mathbf{N}^*$ for every $t \in \mathcal{G}_\omega$ and ξ_t and ξ_s are of different types for $t \neq s$, $t, s \in \mathcal{G}_\omega$. Then the space $\beta\mathcal{G}_\omega$ is rigid.*

PROOF. Let $h\colon \beta\mathcal{G}_\omega \to \beta\mathcal{G}_\omega$ be an autohomeomorphism and put $D = h(\mathcal{G}_\omega) \cap \mathcal{G}_\omega^*$. Then D is a countable subset of $\mathcal{G}_\omega^*$. In virtue of Theorem 1, $\mathrm{cl}\, D \subset \mathcal{G}_\omega^*$ and therefore $\mathrm{cl}\, D$ is a nowhere dense closed subset of $\beta\mathcal{G}_\omega$. Hence $\mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)$ is an open dense subspace of the space $\mathcal{G}_\omega$ and $h(\mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)) \subset \mathcal{G}_\omega$. We shall show that $h(t) = t$ for every $t \in \mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)$.

Let $t \in \mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)$ and let $s = h(t)$. The set $A = \{n \in \mathbf{N}\colon t^\frown n \in \mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)\}$ is in ξ_t and the set $B = \{t^\frown n\colon n \in A\}$ is a discrete subset of $\mathcal{G}_\omega - h^{-1}(\mathrm{cl}\, D)\}$ containing t and no other points in its closure. Hence $h(B)$ is a discrete subset of $\mathcal{G}_\omega$ containing s and no other points in its closure. This implies that $\{n \in \mathbf{N}\colon s^\frown n \in h(B)\} \in \xi_s$. Now consider a map $f\colon A \to \mathbf{N}$ defined as follows: if $n \in A$, then $f(n) = m$, where m is such that $h(t^\frown n) = s^\frown m$. The map f is 1-1. If $\overline{f}$ is the unique extension of f to βA, then $\overline{f}(\xi_t) = \xi_s$, because $h(t) = s$. This shows that the ultrafilters ξ_t and ξ_s are of the same type and in consequence that $t = s = h(t)$. Since the restriction of h to some dense subset of $\beta\mathcal{G}_\omega$ is the identity map, h has to be the identity map as well. □

Within ZFC, K. Kunen [**K**] has shown that there are 2^c different types among weak P-points of $\mathbf{N}^*$. Each selection of countably many such types produces a rigid separable Stone space. Different selections give nonhomeomorphic spaces. So finally we have

MAIN THEOREM. *There exist 2^c pairwise nonhomeomorphic rigid separable Stone spaces.*

ADDENDUM. We were informed by Jan van Mill that E. van Douwen has obtained independently similar results and P. Simon has also known some results concerning the spaces (Seq, $\mathcal{T}$).

REFERENCES

[**AF**] A. V. Arhangelskii and S. P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. **15** (1968), 313–330.

[**DMR**] E. K. van Douwen, J. D. Monk and M. Rubin, *Some questions about Boolean algebras*, Algebra Universalis **11** (1980), 220–243.

[**G**] A. V. Gubbi, *On a class of projective spaces*, Doctoral Dissertation, Memphis State University, Memphis, Tenn., 1984.

[**J**] T. J. Jech, *Set theory*, Academic Press, New York, 1978.

[**K**] K. Kunen, *Weak P-points in* $\mathbf{N}^*$, Colloq. Math. Soc. Janos Bolyai **23** (1980), 741–749.

[**KR**] V. Kannan and M. Rajagopalan, *Constructions and applications of rigid spaces*. I, Advances in Math. **29** (1978), 89–130.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, M5S 1A1, ONTARIO, CANADA (Current address of A. Dow)

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, YOUNGSTOWN STATE UNIVERSITY, YOUNGSTOWN, OHIO 44555

Current address (A. V. Gubbi): Department of Mathematics, Southwest Missouri State University, Springfield, Missouri 65804

DEPARTMENT OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE, POLAND

Current address (A. Szymański): Department of Mathematics, Slippery Rock State University, Slippery Rock, Pennsylvania 16057