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SPACES WHICH ADMIT AR-RESOLUTIONS

A. KOYAMA, S. MARDEŠIĆ AND T. WATANABE

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ABSTRACT. It is proved that a topological space X admits an AR-resolution (in the sense of [6]) if and only if X has trivial (strong) shape.

1. Introduction. It is well known that every topological space X admits an ANR-resolution in the sense of [6], i.e., a resolution $p = (p_a)$: $X \to \mathbf{X} = (X_a, p_{aa'}, A)$, where each X_a is an ANR (for metric spaces) [6, Theorem 12]. A. Koyama raised the question to determine which spaces admit AR-resolutions, i.e., resolutions where each X_a is an AR (for metric spaces). Clearly, such spaces must be of trivial shape. In this note we show that the converse also holds. More precisely, we prove the following

THEOREM. *Let X be a topological space. Then the following three conditions are equivalent:*
(i) *X admits an AR-resolution.*
(ii) *X has trivial strong shape.*
(iii) *X has trivial shape.*

Recall that compact metric spaces of trivial shape are limits of inverse sequences of Hilbert cubes [1, 5].

2. Implication (i) $\Rightarrow$ (ii). We use the notions of coherent map of systems, coherent homotopy and strong shape, ssh, as defined in [2–4].

We first establish a lemma.

LEMMA 1. *Let $\mathbf{X} = (X_a, p_{aa'}, A)$ and $\mathbf{Y} = (Y_b, q_{bb'}, B)$ be inverse systems of metric spaces over (directed) antisymmetric cofinite index sets. If all Y_b are AR's, then any two coherent maps f, f': $\mathbf{X} \to \mathbf{Y}$ are coherently homotopic.*

PROOF. Since B is cofinite, without loss of generality we can assume that f and f' are special coherent maps [4, Lemma 5], i.e., are given by increasing functions φ, φ': $B \to A$ and by maps $f_{\mathbf{b}}$: $\Delta^n \times X_{\varphi(b_n)} \to Y_{b_0}$, $f'_{\mathbf{b}}$: $\Delta^n \times X_{\varphi'(b_n)} \to Y_{b_0}$, respectively; here $\mathbf{b} = (b_0, \dots, b_n)$, $b_0 \leq \cdots \leq b_n$. We choose an increasing function Φ: $B \to A$ such that $\Phi \geq \varphi, \varphi'$. We need maps $F_{\mathbf{b}}$: $\Delta^n \times I \times X_{\Phi(b_n)} \to Y_{b_0}$ such that $(\Phi, F_{\mathbf{b}})$ is a coherent map F: $I \times \mathbf{X} \to \mathbf{Y}$ satisfying

$$F_{\mathbf{b}}(t, 0, x) = f_{\mathbf{b}}(t, p_{\varphi(b_n)\Phi(b_n)}(x)), \tag{1}$$

$$F_{\mathbf{b}}(t, 1, x) = f'_{\mathbf{b}}(t, p_{\varphi'(b_n)\Phi(b_n)}(x)). \tag{2}$$

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We construct the maps $F_{\mathbf{b}} = F_{b_0\cdots b_n}$ by induction on n. Since Y_{b_0} is an AR, the maps

$$f_{b_0}(0 \times p_{\varphi(b_0)\Phi(b_n)})\colon\ 0 \times X_{\Phi(b_0)} \to Y_{b_0}$$

and

$$f'_{b_0}(1 \times p_{\varphi'(b_0)\Phi(b_0)})\colon\ 1 \times X_{\Phi(b_0)} \to Y_{b_0}$$

extend to a map $F_{b_0}\colon I \times X_{\Phi(b_0)} \to Y_{b_0}$.

Now assume that we have already defined maps $F_{\mathbf{b}}$, where $\mathbf{b}$ has $< n$ terms, $n \geq 1$. We will now define $F_{b_0\cdots b_n}\colon \Delta^n \times I \times X_{\Phi(b_n)} \to Y_{b_0}$.

We first consider the case when $(b_0, \ldots, b_n)$ is nondegenerate, i.e., $b_j \neq b_{j+1}$ for all j. By the requirement that F be a coherent map (see [**2**, **4**]) and by (1) and (2), the map $F_{b_0\cdots b_n}$ is already determined on the closed subset $\partial(\Delta^n \times I) \times X_{\Phi(b_n)}$ of $\Delta^n \times I \times X_{\Phi(b_n)}$. Since Y_{b_0} is an AR, extensions to all of $\Delta^n \times I \times X_{\Phi(b_n)}$ exist. We take such an extension for $F_{b_0\cdots b_n}$.

If $(b_0, \ldots, b_n)$ is degenerate, say $b_j = b_{j+1}$, then we put

$$F_{b_0\cdots b_n}(t, x) = F_{b_0\cdots \hat{b}_j \cdots b_n}(\sigma_j t, x), \tag{3}$$

where $(b_0, \ldots, \hat{b}_j, \ldots, b_n)$ is obtained from $(b_0, \ldots, b_n)$ by omitting b_j and $\sigma_j\colon \Delta^n \to \Delta^{n-1}$ is the jth degeneracy operator.

PROOF OF (i) $\Rightarrow$ (ii). By assumption, X admits an AR-resolution $p = (p_a)\colon X \to \mathbf{X} = (X_a, p_{aa'}, A)$. We can always achieve that A is antisymmetric and cofinite (use the construction described in [**7**, I, §1.2, Theorem 2]). On the other hand, a single-point space $Y = \{*\}$ also admits an AR-resolution $q = (q_b)\colon Y \to \mathbf{Y}$ with antisymmetric and cofinite index set, e.g., such is the rudimentary resolution $1\colon Y \to (Y_{b_0})$, where $Y_{b_0} = Y = \{*\}$.

Constant maps $u\colon X \to Y$ and $v\colon Y \to X$ induce coherent maps $f\colon \mathbf{X} \to \mathbf{Y}$ and $g\colon \mathbf{Y} \to \mathbf{X}$ (see [**4**, II, §2]). By Lemma 1, $gf \simeq 1$ and $fg \simeq 1$. Therefore, the induced strong shape morphisms $S_1(u)\colon X \to Y$ and $S_1(v)\colon Y \to X$ satisfy

$$S_1(v)S_1(u) = 1 \quad \text{and} \quad S_1(u)S_1(v) = 1. \tag{4}$$

This shows that the strong shape $\mathrm{ssh}(X) = \mathrm{ssh}(Y)$. Since $Y = \{*\}$, we conclude that $\mathrm{ssh}(X) = 0$.

3. Implication (ii) $\Rightarrow$ (iii). This is an immediate consequence of the fact that there exists a functor $S_2\colon \mathrm{ssh} \to \mathrm{sh}$ from the strong shape category ssh to the shape category sh [**4**, II, §2].

4. Implication (iii) $\Rightarrow$ (i). In the proof we need a simple lemma.

LEMMA 2. *Let $p\colon X \to Y$ be a mapping from a metric space X into an ANR Y. If p is homotopic to a constant map, then there exist an AR Z and a map $r\colon Z \to Y$ such that $X \subseteq Z$ is a closed subset and $r|X = p$.*

PROOF OF LEMMA 2. By the Kuratowski-Woydisławski embedding theorem (see, e.g., [**7**, I, §3.1, Theorem 2], X can be embedded as a closed subset in a convex set Z of a normed vector space. By the Dugundji extension theorem (see, e.g., [**7**, I, §3.1, Theorem 3], Z is an AR. Since p is homotopic to a constant map $c\colon X \to Y$ and constants extend to all of Z, the homotopy extension theorem (see, e.g., [**7**, I, §3.2, Theorem 4] proves that also $p\colon X \to Y$ extends to a map $r\colon Z \to Y$.

PROOF OF (iii) $\Rightarrow$ (i). Choose an antisymmetric cofinite ANR-resolution $p = (p_a)\colon X \to \mathbf{X} = (X_a, p_{aa'}, A)$ for the space X. Since $\operatorname{sh} X = 0$, every $a \in A$ admits an $a' = \varphi(a) \geq a$ such that

$$(1) \qquad P_{a\varphi(a)} \simeq 0$$

(the proof of [**7**, II, §2.3, Theorem 7] applies). Since A is cofinite, we can achieve that $\varphi\colon A \to A$ is an increasing function.

If A contains an element $a_0 \in A$ such that $a_0 \leq a$ implies $a_0 = a$, then (by directedness and antisymmetry) a_0 is the unique element $a_0 = \max A$. In this case $p_{a_0}\colon X \to (X_{a_0})$ is also a resolution and since $\varphi(a_0) = a_0$, (1) shows that X_{a_0} is contractible and therefore an AR, which verifies the assertion.

We will now consider the case where A has no maximal element, i.e., every $a \in A$ admits an $a' \geq a$, which is different from a.

By Lemma 2 and (1), with each $a \in A$ one can associate an AR Z_a and a map $r_a\colon Z_a \to X_a$ such that $X_{\varphi(a)} \subseteq Z_a$ is a closed subset and

$$(2) \qquad r_a | X_{\varphi(a)} = p_{a\varphi(a)}.$$

We now define a new ordering $\leq^*$ in A by putting $a_1 \leq^* a_2$ provided $a_1 = a_2$ or $\varphi(a_1) \leq a_2$. Clearly, the set $A^* = (A, \leq^*)$ is directed, antisymmetric and cofinite. For $a_1 \leq^* a_2$ we define a map $q_{a_1a_2}\colon Z_{a_2} \to Z_{a_1}$ by putting $q_{a_1a_2} = \mathrm{id}$ if $a_1 = a_2$, and

$$(3) \qquad q_{a_1a_2} = p_{\varphi(a_1)a_2} r_{a_2} \quad \text{if } a_1 \leq^* a_2, \quad a_1 \neq a_2.$$

Note that in the latter case

$$(4) \qquad q_{a_1a_2}(Z_{a_2}) \subseteq X_{\varphi(a_1)}.$$

If $a_1 \leq^* a_2 \leq^* a_3$, then

$$q_{a_1a_2} q_{a_2a_3} = q_{a_1a_3}.$$

This is clear if $a_1 = a_2$ or $a_2 = a_3$. In the remaining case we have

$$(6) \qquad q_{a_1a_2} q_{a_2a_3} = p_{\varphi(a_1)a_2} r_{a_2} p_{\varphi(a_2)a_3} r_{a_3} = p_{\varphi(a_1)a_3} r_{a_3} = q_{a_1a_3},$$

because (2) implies

$$(7) \qquad r_{a_2} p_{\varphi(a_2)a_3} = p_{a_2\varphi(a_2)} p_{\varphi(a_2)a_3} = p_{a_2a_3}.$$

We have, thus, proved that $\mathbf{Z} = (Z_a, q_{aa'}, A^*)$ is an AR-system.

We now define maps $q_a\colon X \to Z_a$ by putting

$$(8) \qquad q_a = p_{\varphi(a)}\colon X \to X_{\varphi(a)} \subseteq Z_a.$$

Note that $a_1 \leq^* a_2$ implies

$$(9) \qquad q_{a_1a_2} q_{a_2} = q_{a_1}.$$

This is obvious if $a_1 = a_2$. In the remaining case, we have, by (2),

$$(10) \qquad q_{a_1a_2} q_{a_2} = p_{\varphi(a_1)a_2} r_{a_2} p_{\varphi(a_2)} = p_{\varphi(a_1)} = q_{a_1}.$$

The proof will be completed if we show that $q = (q_a)\colon X \to \mathbf{Z}$ is a resolution, i.e., has properties (R1) and (R2) (see [**6** or **7**]).

PROOF OF (R1). Let P be an ANR, let $\mathcal{U}$ be an open covering of P and $f\colon X \to P$ a mapping. By (R1) for $p\colon X \to \mathbf{X}$ there exists an $a \in A$ and a map $h\colon X_a \to P$ such that hp_a and f are $\mathcal{U}$-near maps, which we denote by

$$(hp_a, f) < \mathcal{U}. \tag{11}$$

Consider the map $h' = hr_a\colon Z_a \to P$. By (2), we have

$$h'q_a = hr_a p_{\varphi(a)} = hp_{a\varphi(a)}p_{\varphi(a)} = hp_a. \tag{12}$$

Consequently, (11) becomes the desired relation

$$(h'q_a, f) < \mathcal{U}. \tag{13}$$

PROOF OF (R2). Let P be an ANR and $\mathcal{U}$ an open covering of P. We choose a covering $\mathcal{U}'$ of P satisfying (R2) for $p\colon X \to \mathbf{X}$. Then $\mathcal{U}'$ also satisfies (R2) for $q\colon X \to \mathbf{Z}$. Indeed, assume that $a \in A' = A$ and $h, h'\colon Z_a \to P$ are maps satisfying

$$(hq_a, h'q_a) < \mathcal{U}'. \tag{14}$$

Since $hq_a = hp_{\varphi(a)}$, $h'q_a = h'p_{\varphi(a)}$, we conclude that there is an $a' \geq \varphi(a)$ such that

$$(hp_{\varphi(a)a'}, h'p_{\varphi(a)a'}) < \mathcal{U}. \tag{15}$$

Note that $a \leq^* a'$ so that $q_{aa'}$ is defined. By the assumption on A, we can choose $a' \geq \varphi(a)$ such that $a' \neq \varphi(a)$ and therefore, $a' \neq a$. It follows that $q_{aa'} = p_{\varphi(a)a'}r_{a'}$ and therefore (15) becomes the desired relation

$$(hq_{aa'}, h'q_{aa'}) < \mathcal{U}. \tag{16}$$

This completes the proof of the Theorem.

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DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, IKEDA CITY, OSAKA 563, JAPAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, 41001 ZAGREB, P.O.B 187, YUGOSLAVIA

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, UNIVERSITY OF YAMAGUCHI, YAMAGUCHI, JAPAN