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A CONJECTURE OF S. CHOWLA VIA THE GENERALIZED RIEMANN HYPOTHESIS

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ABSTRACT. S. Chowla conjectured that if $p = m^2 + 1$ is prime and $m > 26$, then h_K, the class number of $K = Q(\sqrt{p})$, is greater than 1. We prove this conjecture under the assumption of the Riemann hypothesis for ζ_K, the zeta function of K, i.e. the generalized Riemann hypothesis (GRH).

It is the purpose of this note to prove the following result.

THEOREM. *Let $K = Q(\sqrt{p})$, where $p = m^2 + 1$ is prime and $m > 26$. If the Riemann hypothesis holds for ζ_K then $h_K > 1$.*

Without the GRH hypothesis, this is known as the Chowla conjecture given in [**1**]. We note that it is an easy consequence of the celebrated Brauer-Siegel theorem that there are only finitely many such p for which $h_K = 1$. In [**3**] Mollin reduced the problem to the case where $m = 2r$ and $r > 13$ is prime.

In what follows we make use of an idea of Cornell and Washington [**2**] to show how to use the GRH to get an effective bound ($p > 10^{23}$) for which the Chowla conjecture holds. The remaining finite cases are then handled by a simple sieve process. Throughout the remainder of the paper p will denote a prime of the form $4r^2 + 1$ where r is an odd prime, and K will denote $Q(\sqrt{p})$.

The following result contains facts which are either well known or trivial. Therefore we state it without proof.

LEMMA. *Let y be a real number.*
(1) *If $y > 1$, then $\sum 1/q < \log y$, where the sum ranges over all primes $q \leq y$.*
(2) *If $y > 0$, then $\sum(1/q^2) < 1/y$, where the sum ranges over all primes $q > y$,*
(3) *If $|y| \leq 1/2$, then $|\log(1-y) + y| < y^2$.*

Now we are in a position to prove the Theorem.

Let $\varepsilon(q) = (p/q)$ denote the Kronecker symbol where q is a prime. Set

$$T_1(y) = \sum_{q \leq y} [q/(q - \varepsilon(q))] \quad \text{and} \quad T_2(y) = \prod_{q > y} [q/(q - \varepsilon(q))].$$

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Therefore $L(1,\chi) = T_1(y)T_2(y)$, the Euler product for the Dirichlet L-function. Clearly we have that $T_1(y) \geq \prod_{q\leq y}(q/(q+1))$. Therefore

(i) $$\log T_1(y) \geq -\sum_{q\leq y}\log(q+1/q) > -\sum_{q\leq y} 1/q > -\log y,$$

where the last inequality is from Lemma (1). Furthermore, we have from Lemma, (2)–(3), that

(ii) $$\log T_2(y) = -\sum_{q>y}\log(1-(\varepsilon(q)/q)) > \sum_{q>y}\varepsilon(q)/q - 1/y.$$

By similar reasoning to that used by Cornell and Washington [**2**, p. 265] (where the GRH is assumed) we get

(iii) $$\sum_{q>y}\varepsilon(q)/q \geq -B(y)[(4+3\log y)/\sqrt{y}],$$

where $B(y) = (\log p)((1/\pi\log y) + (5.3/(\log y)^2)) + 4/\log y + 1/\pi$. Now, from (ii) and (iii), we get

(iv) $$\log T_2(y) > -B(y)[(4+3\log y)/\sqrt{p}] - 1/y.$$

Hence from (i) and (iv) we have

$$L(1,\chi) = T_1(y)T_2(y) > (1/y)\exp[-B(y)[(4+3\log y)/\sqrt{p}] - 1/y].$$

Set $y = (\log p)^2$; whence $\log y = 2\log\log p$. Thus

$$\begin{aligned} &-B(y)[(4+3\log y)/\sqrt{y}] - 1/y \\ &= -\{2/(\pi\log\log p) + 15/(\log\log p)^2 + 8/(\log\log p)\log p + 4/(\pi\log p) \\ &\qquad + 3/\pi + 135/(2\log\log p) + 12/\log p + 6\log\log p/(\pi\log p) + 1/(\log p)^2\} \\ &= d(p), \end{aligned}$$

say. Hence $L(1,\chi) > e^{d(p)}/(\log p)^2$. However, $2h_K R = \sqrt{p}L(1,\chi)$, where $R = \log(2r+\sqrt{p}) < \log 2\sqrt{p}$ is the regulator of K; whence

$$h_K > \frac{\sqrt{p}e^{d(p)}}{2(\log p)^2(\log 2\sqrt{p})} = F(p),$$

say. Moreover $-d(p)$ is a decreasing function of p so then $e^{d(p)}$ is an increasing function of p, forcing $F(p)$ to be an increasing function of p. Thus, if $p > 10^{13}$ then $d(p) < 4.68$, $e^{d(p)} > 0.009279493$ and $\sqrt{p}/(2(\log p)^2(\log 2\sqrt{p})) \geq 112.6838154$. Hence $F(p) > 1$ for $p > 10^{13}$.

Now, to deal with the primes $p < 10^{13}$ where $p = 4r^2+1$ we note that $r < 10^{6\cdot 5}/2 \approx 1.6\times 10^6$. Hereafter, we use the fact proved by Mollin in [**3**] that $h_K = 1$ is tantamount to all primes less than r being inert in K. Select some positive integer k and the first k primes $\{q_i\}_{i=1}^k$ with $q_1 = 5$. For each of these q_i find (by trial) and tabulate those S_{ij} such that $0 \leq S_{ij} \leq q_i - 1$ and $((4S_{ij}^2+1)/q_i) \neq -1$ where $(\ /\ )$ is the Legendre symbol. There are approximately $q_i/2$ of these. If any $r \equiv S_{ij} \pmod{q_i}$ and $r > q_i$ then delete this value of r since either: $4r^2+1 \equiv 0 \pmod{q_i}$, which means that $4r^2+1$ is not prime, or $((4r^2+1)/q_i) = 1$ and $q_i < r$, which means that $h_K > 1$. Since half of the r's are eliminated for each q_i, then a value of k such that $2^k > 1.6 \times 10^6$ should suffice to complete the task. We used $k = 100$ and a Fortran program to sieve out as many values of $r < 1.6 \times 10^6$ as possible. In a matter of a few minutes we found that if $r > 13$ and $p = 4r^2 + 1$ is a prime ($p < 10^{13}$) there exists some $q_i < r$ such that q_i is quadratic residue of p. This proves the Theorem. Q.E.D.

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